ETH/Uni Zürich — Alpbach summer school 2015
Relation to Eisenstein series

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Contents

1 Introduction

The goal of these notes is to compute the Fourier coefficients of the following Eisenstein series and to present their relation with the Weil representation.
Let $q \equiv 3$ modulo 4 a prime and $k = \mathbb{Q}(\sqrt{-q})$. Let

$$\begin{aligned} \chi : \mathbb{Q}^* \backslash \mathbb{A}^* &\to \{\pm 1\} \\ a &\mapsto (a, -q)_{\mathbb{A}} := \prod_{\nu} (a, -q)_{\nu} \end{aligned}$$

be the associated quadratic character where $(x,q)_\nu$ is the local Hilbert symbol.
Write $\tau = \mu + \nu i \in \mathbb{H}$. There are 2 Eisenstein series of weight 1 associated with k:

$$E_\pm(\tau;s) = \nu^{\frac{s}{2}} \sum_{\sigma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} (c\tau+d)^{-1}|c\tau+d|^{-s}\Phi_q^\pm(\sigma)$$

where for $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$\Phi_g^\pm(\sigma) = \begin{cases} \chi_q(a) & \text{if } c \equiv 0 \text{ modulo } q, \\ \pm i q^{-1/2}\chi_q(c) & \text{if } c \not\equiv 0 \text{ modulo } q. \end{cases}$$

Here $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \mid x \in \mathbb{Z}\}$. The Eisenstein series is a modular form of weight 1 w.r.t. the group $\Gamma_0(q)$ and character $\chi_q(a)$ lifted to $\Gamma_0(q)$.
The series converge for $\Re(s) > 1$ and have an analytic continuation to the whole complex plane. The normalized series

$$E_\pm^*(\tau;s) := q^{\frac{s+1}{2}}\Lambda(s+1,\chi)E_\pm(\tau;s)$$

where

$$\Lambda(s,\chi) = \pi^{\frac{s+1}{2}}\Gamma(\frac{s+1}{2})L(s,\chi)$$

satisfy the functional equation

$$E_\pm^*(\tau;-s) = \pm E_\pm^*(\tau;s)$$

From the functional equation follows that E_- vanishes identically for $s=0$. We will explain this in more detain the sequel of these notes.
Actually $E_+^*(\tau;0)$ is holomorphic and we have

$$E_+^*(\tau;0) = 2h_k + 4\sum_{n=1}^\infty \rho(n)e(n\tau) \tag{1}$$

where $\rho(n)$ is the number of integral ideals of norm n. This is a special case of the Siegel-Weil formula that we will explain in section 6.1.
We are interested in the derivative $\frac{\mathrm{d}}{\mathrm{d}\,s}E_-^*(\tau,s)|_{s=0}$. One purpose of these notes is to compute its Fourier expansion. The result is:

Theorem 1.1.

$$\frac{\mathrm{d}}{\mathrm{d}\,s}E_-^*(\tau,s)|_{s=0} = \sum_{t\in\mathbb{Z}} a_t(\nu)e(t\tau)$$

where

$$a_t(\nu) = \begin{cases} -e_p\log(p)(\mathrm{ord}_p(t)+1)\rho(tp^{e_p-2})) & \textit{if } t>0, \mathrm{Diff}(t) = \{p\}, \\ 0 & \textit{if } t>0, |\mathrm{Diff}(t)|>1, \\ -h_k\left(\log(q)+\log(\nu)+2\frac{\Lambda'(1,\chi)}{\Lambda(1,\chi)}\right) & \textit{if } t=0, \\ -2\mathrm{Ei}(4\pi t\nu)\rho(-t) & \textit{if } t<0. \end{cases}$$

with

$$\beta_1(t) = \int_1^\infty u^{-1} e^{-ut} \, \mathrm{d}\, u$$

and

$$\mathrm{Diff}(t) = \{p \mid \chi_p(-t) = -1\}$$

Here e_p is the ramification index of p in k, i.e. 2 if $p = q$ and 1 otherwise.
Note that the positive Fourier coefficients are holomorphic (i.e. do not depend on ν).

2 Adelic versus classical modular forms

To understand the Eisenstein series, it will be convenient to work adelically. Let us briefly recall the translation of classical modular forms to adelic automorphic forms.
Write $K_\infty \cong S^1$ for the subgroup of $\mathrm{SL}_2(\mathbb{R})$ of elements $k_{a+bi} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ with $a^2 + b^2 = 1$.
Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. Recall that a modular form of weight $k \in \mathbb{Z}$ w.r.t. Γ is a function $f : \mathbb{H} \to \mathbb{C}$ transforming as

$$f(g\tau) = j(g, \tau)^k f(\tau)$$

for all $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $j(g, \tau) = c\tau + d$. Usually it has to satisfy a growth condition at infinity, which we will not state here.
Now let $\mathbb{A} = \prod'_\nu \mathbb{Q}_\nu$ be the ring of adeles for $\mathbb{Q}$ and $\mathrm{SL}_2(\mathbb{A})$ the group of adelic 2x2-matrices of determinant 1. We sometimes write $\mathbb{A}_f$ for the finite adeles (in which the factor $\mathbb{R}$ is omitted). First, since Γ is a congruence subgroup, we find a compact open subgroup K_f of $\mathrm{SL}_2(\mathbb{A}_f)$ such that $K_f \cap \mathrm{SL}_2(\mathbb{Q}) = \Gamma$. Then observe that by strong approximation (i.e. $\mathrm{SL}_2(\mathbb{Q})$ lies dense in $\mathrm{SL}_2(\mathbb{A}_f)$[1]), we have:

$$\mathrm{SL}_2(\mathbb{A}) = \mathrm{SL}_2(\mathbb{Q}) \, \mathrm{SL}_2(\mathbb{R}) K_f$$

Recall that there is a bijection between smooth (non-holomorphic) modular forms $f : \mathbb{H} \to \mathbb{C}$ for Γ of weight k and smooth functions (see [1] for the definition of smooth in this context)

$$\phi_f : \mathrm{SL}_2(\mathbb{Q}) \backslash \mathrm{SL}_2(\mathbb{A}) \to \mathbb{C}$$

with the property

$$\phi_f(g k_f k_\alpha) = \alpha^k \phi_f(g)$$

for $k_\alpha \in K_\infty, k_f \in K_f$, given as follows:

$$\phi_f(g_{\mathbb{Q}} g_\infty k_f) := f(g_\infty \cdot i) j(g_\infty, i)^{-k}$$

[1]Sketch of proof. Reduce to show that $\mathrm{SL}_2(\mathbb{Z})$ lies dense in $\mathrm{SL}_2(\widehat{\mathbb{Z}})$. This follows, if we can show that $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is surjective because $\mathrm{SL}_2(\widehat{\mathbb{Z}}) = \lim \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Then observe that $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is generated by elementary matrices. Using the Chinese remainder theorem it suffices to show this for $\mathrm{SL}_2(\mathbb{Z}/p^n\mathbb{Z})$, where Gaussian elimination works. Elementary matrices clearly lift to $\mathrm{SL}_2(\mathbb{Z})$.

f can be reconstructed as

$$f(\tau) = \phi(g_\infty 1_f) j(g_\infty, i)^k$$

where $\tau = \mu + \nu i$ and $g_\infty \in \mathrm{SL}_2(\mathbb{R})$ is any element with $g_\infty i = \tau$, for example

$$f(\tau) = \phi(\begin{pmatrix} \nu^{1/2} & \mu\nu^{-1/2} \\ & \nu^{-1/2} \end{pmatrix}_\infty 1_f)\nu^{-k/2}.$$

One can match the growth conditions at cusps in both languages, which we will not do here. If $\chi : \Gamma \to S^1$ is a character that is trivial on another congruence subgroup satisfying $\Gamma' = K'_f \cap \mathrm{SL}_2(\mathbb{Q})$, we may regard χ as a character of $\chi : K_f \to S^1$ via the isomorphism $K_f/K'_f \cong \Gamma/\Gamma'$. Then modular forms f which transform via χ under Γ correspond precisely to functions $\phi_f : \mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A}) \to \mathbb{C}$ which satisfy:

$$\phi_f(gk_fk_\alpha) = \alpha^k \chi(k_f)\phi_f(g)$$

for $k_f \in K_f$.

Example 2.1. *Let*

$$K_0(q) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\widehat{\mathbb{Z}}) \mid c \equiv 0 \textit{ modulo } q\}$$

with

$$\Gamma_0(q) = K_0(q) \cap \mathrm{SL}_2(\mathbb{Q}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \textit{ modulo } q\}.$$

We have the following character

$$\begin{aligned} \chi : K_0(q) &\to S^1 \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto \chi_q(a) \end{aligned}$$

Note that a cannot be divisible by q. Here $\chi_q(a)$ was defined in section 1.

3 Parabolic induction for SL_2 and Eisenstein series.

We will now explain the general theory of Eisenstein series for SL_2 in the adelic language. Let $\chi = \prod_\nu \chi_\nu : \mathbb{A}^*/\mathbb{Q}^* \to S^1$ be a continuous character and assume that χ_∞ has values in ± 1.

Consider the group $M \subset \mathrm{SL}_2$ consisting of diagonal matrices and the group N of unipotent upper triangular matrices. We denote $B = M \cdot N$ the group of upper triangular matrices which is a Borel subgroup.

We consider the character

$$\chi |\cdot|_{\mathbb{A}}^s : \mathbb{A}^*/\mathbb{Q}^* \to \mathbb{C}^*.$$

It may be extended to $B(\mathbb{A})$ using the decomposition $B(\mathbb{A}) = M(\mathbb{A})N(\mathbb{A})$.

Let $I(\chi|\cdot|_{\mathbb{A}}^s)$ be the normalized induced representation from $B(\mathbb{A})$ to $\mathrm{SL}_2(\mathbb{A})$, i.e.

$$I(\chi|\cdot|^s) = \left\{ \begin{array}{c} f : \mathrm{SL}_2(\mathbb{A}) \to \mathbb{C} \text{ smooth, } K_\infty\text{-finite, satisfying} \\ f(\begin{pmatrix} \alpha & x \\ & \alpha^{-1} \end{pmatrix} g) = |\alpha|^{s+1}\chi(\alpha)f(g) \end{array} \right\}$$

(the $+1$ comes from the normalization). Likewise, we define $I_{\mathbb{R}}(\chi_\infty|\cdot|^s)$ and $I_{\mathbb{Q}_p}(\chi_p|\cdot|_p^s)$. For almost all p, the character χ_p is trivial on $\mathbb{Z}_p^*$ and hence the function

$$\xi_p^0(\begin{pmatrix} \alpha & x \\ & \alpha^{-1} \end{pmatrix} k) = |\alpha|_p^{s+1}\chi(\alpha)$$

for $k \in \mathrm{SL}_2(\mathbb{Z}_p)$, defined using the Iwasawa decomposition, is well-defined. It follows that

$$I(\chi|\cdot|_{\mathbb{A}}^s) = \bigotimes_\nu{}' I_\nu(\chi_\nu|\cdot|_\nu)$$

where the restricted tensor product is formed w.r.t. the vectors ξ_p^0.
We have the following vectors in $I_\infty(\chi_\infty|\cdot|^s)$

$$\xi_\infty^k(\begin{pmatrix} \alpha & x \\ & \alpha^{-1} \end{pmatrix} k_z) = z^k|\alpha|^{s+1}\chi_\infty(\alpha).$$

Here either k is even and χ_∞ trivial, or k is odd and χ_∞ is the sign character. These vectors form a basis of $I_\infty(\chi_\infty|\cdot|^s)$. We will see in section 5 how it decomposes into irreducibles. The theory of Eisenstein series realizes the $\mathrm{SL}_2(\mathbb{A})$-principal series representations ('representation' to be understood in the appropriate sense) $I_{\mathbb{A}}(\chi|\cdot|_{\mathbb{A}}^s)$ in the space of automorphic forms.

3.1. Let

$$\Phi(s) \in I_{\mathbb{A}}(\chi|\cdot|_{\mathbb{A}}^s)$$

be such that its restriction to $K = K_f K_\infty$ is independent of s. It follows from the above that such a section is uniquely determined by its value $\phi(0)$ and is given by

$$\Phi(s)(\begin{pmatrix} \alpha & x \\ & \alpha^{-1} \end{pmatrix} k) = |\alpha|^{s+1}\,\chi(\alpha)\,\Phi(0)(k),$$

where $k \in K_\infty \mathrm{SL}_2(\widehat{\mathbb{Z}})$. Given such a section, we define the Eisenstein series by

$$E_\Phi(g;s) := \sum_{\gamma \in B(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{Q})} \Phi(s)(\gamma g).$$

One can show that it converges to a smooth K_∞-finite function in g, if $\Re(s) > 1$, which depends holomorphically on s. It is then obviously left invariant under $\mathrm{SL}_2(\mathbb{Q})$.
Therefore it defines a morphism of $\mathrm{SL}_2(\mathbb{A})$-representations:

$$\begin{array}{rcl} E : I_{\mathbb{A}}(\chi|\cdot|_{\mathbb{A}}^s) & \to & \mathcal{A}(\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A})) \\ \Phi(s) & \mapsto & E(\Phi(s), -) \end{array}$$

The spaces $I_R(\chi)$ can be understood in terms of Weil representations. We will introduce them in the next section.

4 The Weil representation

4.1. Let R be a locally compact topological ring in which 2 is invertible. In the following R will be one of $\mathbb{Q}_p, \mathbb{R}, \mathbb{A}, \mathbb{A}_f, \mathbb{F}_p$.
Choose a continous additive character $\Psi : R \to S^1$ such that R becomes self-dual w.r.t. the bicharacter

$$x, y \mapsto \Psi(xy).$$

Choose a self-dual measure $\mathrm{d}\, x$ on R w.r.t. this character. On $L^2(R)$ we have the structure of a Hilbert space with Hermitian product given by:

$$\langle f, g \rangle = \int_R f(x)\overline{g(x)}\, \mathrm{d}\, x.$$

The Fourier transform of a function $f \in L^2(R)$

$$\widehat{f}(x) := \int_R f(y)\Psi(-xy)\, \mathrm{d}\, y$$

defines a unitary operator in $\mathrm{Aut}(L^2(R))$ and we have

$$\widehat{\widehat{f}}(x) = f(-x).$$

Let V now a free rank n module over R with a bilinear form inducing an isomorphism $V \to V^\vee$. We choose a measure $\mathrm{d}\, x$ on V, too, which is self-dual w.r.t. the bilinear form on V. The Fourier transform of a function $f \in L^2(V)$

$$\widehat{f}(x) := \int_R f(y)\Psi(-x \cdot y)\, \mathrm{d}\, y$$

where $x \cdot y$ is the bilinear form on V therefore also satisfies

$$\widehat{\widehat{f}}(x) = f(-x).$$

Lemma 4.2. *If $R = \mathbb{Q}_p$ and let $M \subset V_{\mathbb{Q}_p}$ be a $\mathbb{Z}_p$-lattice. Denote φ_M its characteristic function. We have*

$$\widehat{\varphi_M} = |M|\varphi_{M^\vee}.$$

Here $|M|$ is the volume of M w.r.t. the self-dual measure. If the bilinear form on V w.r.t. a basis of M has matrix S, we have

$$|M| = (|\det(S)|_p)^{\frac{n}{2}}.$$

4.3. Define the Heissenberg group $H(V) := S_1 \times V \times V$, where $S^1 = \{z \in \mathbb{C} \mid |x| = 1\}$ with multiplication

$$(t_1, v_1, w_1)(t_2, v_2, w_2) = (t_1 t_2 \Psi(v_1 \cdot w_2), v_1 + v_2, w_1 + w_2).$$

We let SL_2 act on V^2 *from the right* as follows:

$$(v,w)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (av+cw, bv+dw).$$

$\mathrm{SO}(V)$ also acts and the two operations commute.
To each $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(R)$ we associate the following quadratic form on $V \times V$ given by

$$Q_\sigma(v,w) := \frac{1}{2}((av+cw)\cdot(bv+dw) - v\cdot w)$$

Let $\mathrm{Aut}^0(H(V))$ be the subgroup of automorphisms of $H(V)$ that fix S^1 and commute with the action of $\mathrm{SO}(V)$. We have the exact sequence

$$0 \longrightarrow V \times V \longrightarrow \mathrm{Aut}^0(H(V)) \longrightarrow \mathrm{SL}_2(R) \longrightarrow 0$$

which is the decomposition into outer and inner automorphisms of $\mathrm{Aut}^0(H(V))$ (everything acts on the right).
The association:

$$(t,v,w)\cdot\sigma \mapsto (t\Psi(Q_\sigma(v,w)), av+cw, bv+dw)$$

defines a splitting $\mathrm{SL}_2(R) \to \mathrm{Aut}^0(H(V))$ of that sequence.
We define a representation of $H(V)$ on $L^2(V)$ by the formula

$$((t,v,w)\varphi)(x) = t\psi(w\cdot x)\varphi(x+v)$$

It is clear that this is a unitary representation.

4.4. The **Stone-von Neumann theorem** says that $H(V)$ is (up to topological issues) the only irreducible unitary representation of $H(V)$ such that S^1 acts naturally. Therefore the map

$$\begin{array}{rcl} N_{\mathrm{Aut}(L_2(V))}(H(V)) & \to & \mathrm{Aut}(H(V)) \\ \widetilde{\sigma} & \mapsto & \widetilde{\sigma}^{-1}h\widetilde{\sigma} \end{array}$$

has to be surjective. Actually we have an exact sequence

$$0 \longrightarrow S_1 \longrightarrow N_{\mathrm{Aut}(L^2(V))}(H(V)) \longrightarrow \mathrm{Aut}(H(V)) \longrightarrow 0 \tag{2}$$

because the representation of $H(V)$ is irreducible.
An element $\widetilde{\sigma}$ is a lift of σ if

$$h\,\widetilde{\sigma}\,\varphi = \widetilde{\sigma}\,(h\cdot\sigma)\,\varphi \tag{3}$$

for any $\varphi \in L^2(V)$.
Consider the following elements

$$m(a) := \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \qquad n(b) := \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \qquad w := \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

The group of all matrices of the form $m(a)$ (the diagonal torus) will be denoted by M, the group of matrices of the form $n(b)$ by N. Their product will be denoted by B.
We guess lifts of these elements to $\mathrm{Aut}(L_2(V))$:

$$\begin{aligned}(\mathbf{m}(a)\varphi)(x) &:= |a|^{\frac{n}{2}}\varphi(ax)\\ (\mathbf{n}(b)\varphi)(x) &:= \Psi(\frac{b}{2}x^2)\varphi(x)\\ \mathbf{w}\varphi(x) &:= \widehat{\varphi}(-x)\end{aligned}$$

where n is the dimension of V. The factor $|a|^{\frac{n}{2}}$ has been introduced to make the operator $\mathbf{m}(a)$ unitary. To justify these formulas, one only has to check that equation (3) holds true for these choices. This is an easy exercise.
These elements fulfill the obvious relations:

$$\mathbf{n}(b)\,\mathbf{m}(a) = \mathbf{m}(a)\,\mathbf{n}(ba^{-2})$$

$$\mathbf{w}\,\mathbf{m}(a) = \mathbf{m}(a^{-1})\,\mathbf{w}$$

4.5. Assume now that R is a field. Then we have the Bruhat decomposition

$$\mathrm{SL}_2(R) = BwB \cup B$$

and the set BwB is precisely the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where c is not zero. Each element $\sigma \in BwB$ has a unique representation in the product NwB, namely:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = n(\frac{a}{c})\,w\,m(-c)\,n(\frac{d}{c}).$$

Hence we may define a lift on BwB by setting:

$$r(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) := \mathbf{n}(\frac{a}{c})\,\mathbf{w}\,\mathbf{m}(-c)\,\mathbf{n}(\frac{d}{c}). \tag{4}$$

A simple calculation shows

$$(r(\begin{pmatrix} a & b \\ c & d \end{pmatrix})\varphi)(y) = |c|^{\frac{n}{2}} \int_V \varphi(ay+cx)\psi(Q_\sigma(y,x))\,\mathrm{d}\,x. \tag{5}$$

Now, if we have a relation $\sigma_1\sigma_2 = \sigma_3$ between these elements, we get from sequence (2) that

$$r(\sigma_1)r(\sigma_2) = \gamma(\sigma_1,\sigma_2)r(\sigma_3)$$

for an element $\gamma(\sigma_1,\sigma_2) \in S^1$.

Proposition 4.6. *$\gamma(\sigma_1,\sigma_2)$ depends only on $c_1^{-1}c_3c_2^{-1} \in R$ which we denote $\gamma(c_1^{-1}c_3c_2^{-1})$. $\gamma(\mu)$ is determined by the equation:*

$$\widehat{f_\mu} = \gamma(\mu)|\mu|^{-\frac{n}{2}} f_{-\mu^{-1}} \tag{6}$$

where the Fourier transform has to be interpreted in the sense of distributions. Here $f_\mu(x) := \Psi(\frac{\mu}{2}x^2)$.

Proof. Inserting the definition (4) and multiplying from the left by elements of the form $\mathbf{m}(a)$ and $\mathbf{n}(b)$, we are left to determine the γ-factor:

$$\mathbf{w}\,\mathbf{n}(\frac{d_1}{c_1})\,\mathbf{n}(\frac{a_2}{c_2})\,\mathbf{w} = \gamma(\sigma_1,\sigma_2)\,r(\sigma_3')$$

in other words $\gamma(\sigma_1,\sigma_2)$ depends only on $\frac{d_1}{c_1}+\frac{a_2}{c_2} = c_1^{-1}c_3c_2^{-1}$ which we denote by $\gamma(\frac{d_1}{c_1}+\frac{a_2}{c_2})$ accordingly.
We therefore have to investigate the relation

$$\mathbf{w}\,\mathbf{n}(\mu)\,\mathbf{w} = \gamma(\mu)\,r(\begin{pmatrix}-1 & \\ \mu & -1\end{pmatrix})$$

We apply both sides to a function φ and evaluate at 0. The left hand side gives

$$\widehat{f_\mu\cdot\widehat{\varphi}}(0) = \langle f_\mu, \widehat{\overline{\varphi}}\rangle$$

for the function $f_\mu(x) = \Psi(\frac{\mu}{2}x^2)$ (note that this is an even function). The right hand side given, using formula (5):

$$\gamma|\mu|^{\frac{n}{2}}\int_V \varphi(\mu x)\,\psi(Q_\sigma(0,x))\,\mathrm{d}\,x$$

where $Q_\sigma(v,w) = \frac{1}{2}((-v+\mu w)\cdot(-w) - v\cdot w) = -\frac{\mu}{2}w^2$.

$$= \gamma|\mu|^{-\frac{n}{2}}\int_V \varphi(x)\,\psi(\frac{-\mu^{-1}}{2}x^2)\,\mathrm{d}\,x = \gamma|\mu|^{-\frac{n}{2}}\,\langle f_{-\mu^{-1}}, \overline{\varphi}\rangle$$

Hence

$$\langle f_\mu, \widehat{\overline{\varphi}}\rangle = \gamma|\mu|^{-\frac{n}{2}}\langle f_{-\mu^{-1}}, \overline{\varphi}\rangle$$

□

4.7. We fix with the following natural (up to a choice of $i\in\mathbb{C}$) character on R:

- $\mathbb{R}$: $\Psi_\infty(x) := e^{2\pi i x}$,
- $\mathbb{Q}_p$: $\Psi_p(x) := e^{-2\pi i[x]}$, where $[x] = \sum_{i<0} x_i p^{-i}$ is the principal part, (it has level/conductor 1),
- $\mathbb{A}$: $\Psi = \prod_\nu \Psi_\nu$.

Theorem 4.8. *If* $\dim(V)$ *is* even, *there is a function* $\rho: BwB \to S^1$ *such that setting for* $\sigma\in BwB$*:*

$$\omega(\sigma) := \rho(\sigma)\,r(\sigma)$$

we have:

1. *For an equation* $\sigma_1\sigma_2 = \sigma_3$ *in* BwB*, the equation*

$$\omega(\sigma_1)\,\omega(\sigma_2) = \omega(\sigma_3)$$

   holds true

2. *ω extends uniquely to a unitary representation $\mathrm{SL}_2(R) \to \mathrm{Aut}(L^2(V_R))$. It fixes the Schwartz-space $S(V_R)$ (i.e. smooth functions of rapid decay, if $R = \infty$, and locally constant with compact support, if $R = \mathbb{Q}_p$).*

3. *We have the following explicit formulas:*

$$\begin{aligned} (\omega(m(a))\varphi)(x) &= \chi_V(a)|a|^{\frac{n}{2}}\varphi(ax) && (7)\\ (\omega(n(b))\varphi)(x) &= \Psi(\frac{b}{2}x^2)\varphi(x) && (8)\\ (\omega(w)\varphi)(x) &= \widetilde{\gamma}(Q)\,\widehat{\varphi}(-x) && (9)\end{aligned}$$

where $Q(x) = \frac{1}{2}x^2$ is the quadratic form on V, and $\chi_V(a) = (a, D)_R$, where D is the discriminant.

The representation ω is called the **Weil representation** associated with V and Ψ.

Proof. 2. follows from 1. because Weil [4, Lemme 6] shows by elementary group theoretic arguments that $\mathrm{SL}_2(R)$ is generated abstractly by BwB and the obvious relations $\sigma_1\sigma_2 = \sigma_3$ for $\sigma_j \in BwB$. The main reason is that BwB forms an *open* algebraic subvariety of SL_2.
To prove 1. we will define $\rho(\sigma) = \rho(c)$ in such a way that the equation

$$\rho(c_1)\,\rho(c_2) = \gamma(c_1^{-1}c_3c_2^{-1})\,\rho(c_3) \tag{10}$$

holds true. 1. then follows from the definition of γ.
The defining equation for $\gamma(\mu)$ shows that it depends only on Ψ and the quadratic form $Q : x \mapsto \frac{\mu}{2}x^2$ that is involved in the definition of f_μ Hence we consider now an arbitrary finite free module V over R and consider the usual Fourier transform $L^2(V) \to L^2(V^\vee)$. Then for any non-degenerate quadratic form Q we may define $\widetilde{\gamma}(Q)$ as the elememt in S^1 such that

$$\widehat{\Psi(Q)} \sim (y \mapsto \widetilde{\gamma}(Q)\Psi(-Q(B^{-1}y)))$$

holds, where $\sim$ means equality up to a positive real scalar and $B : V \to V^\vee$ is the symmetric morphism associated with Q. In other words, we have $\gamma(\mu) = \widetilde{\gamma}(\mu Q)$ in the case considered in this section.
Consider a relation $\sigma = \sigma'\sigma''$ in BwB. Then by definition, denoting Q the quadratic form $x \mapsto \frac{1}{2}x^2$, we have:

$$r(\sigma')r(\sigma'') = \widetilde{\gamma}(\frac{c}{c'c''}Q)r(\sigma).$$

Defining $\omega(\sigma) := (c, D)\,\widetilde{\gamma}(Q)^{-1}\,r(\sigma)$ we get using Lemma 4.9, 6.:

$$\omega(\sigma')\,\omega(\sigma'') = (c', D)\,\widetilde{\gamma}(Q)^{-1}\,(c'', D)\,\widetilde{\gamma}(Q)^{-1}\,\widetilde{\gamma}(\frac{c}{c'c''}Q)\,r(\sigma)$$

$$= (c', D)\,\widetilde{\gamma}(Q)^{-1}\,(c'', D)\,\widetilde{\gamma}(Q)^{-1}\,(\frac{c}{c'c''}, D)\,\widetilde{\gamma}(Q)\,r(\sigma) = (c, D)\,\widetilde{\gamma}(Q)^{-1}\,r(\sigma) = \omega(\sigma).$$

To get the formulas stated in the Theorem note that $(-1, D)\widetilde{\gamma}(Q)^{-1} = \widetilde{\gamma}(Q)$. □

Choose a basis and write this quadratic form as $Q(x) = \sum_i \alpha_i x_i^2$. Then we write also $\widetilde{\gamma}(\alpha_1, \ldots, \alpha_n)$ for $\widetilde{\gamma}(Q)$. We write D (discriminant) for $(-1)^{n-1}\prod_i \alpha_i$. Let $\varepsilon(Q) = \prod_{i<j}(a_i, a_j)_R$ be the Hasse invariant of Q.

Lemma 4.9. *For arbitrary R (in our list):*

1. $\widetilde{\gamma}(\alpha_1, \ldots, \alpha_n) = \prod_i \widetilde{\gamma}(\alpha_i)$.
2. $\widetilde{\gamma}(\alpha)$ *depends only on* α *modulo squares.*
3. $\widetilde{\gamma}(1, -1) = 1$ *or, if* $R = \mathbb{Q}_p$ *and* V *contains a unimodular lattice for* Q*, then* $\widetilde{\gamma}(Q) = 1$.
4. $\widetilde{\gamma}(1)\,\widetilde{\gamma}(-a)\,\widetilde{\gamma}(-b)\,\widetilde{\gamma}(ab) \;=\; (a, b)_R$.
5. $\widetilde{\gamma}_\nu(Q) \;=\; \widetilde{\gamma}_\nu(1)^{n-1}\,\widetilde{\gamma}_\nu((-1)^{n-1}D)\,\varepsilon(Q)$.
6. *If* n *is even, we have*
$$\widetilde{\gamma}(\mu Q) = (\mu, D)_R \widetilde{\gamma}(Q).$$

Proof. 1. and 2. follow directly from the definiton.
3. and 4. are proven in Weil's article [4].
5. and 6. follow by induction from 3. and 4. ☐

We state here even more explicit calculations:

Lemma 4.10. *Let* $\rho = \exp(\frac{2\pi i}{8})$.

1. *For* p *odd:*
$$\widetilde{\gamma}_p(\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ is a unit,} \\ p^{-\frac{1}{2}} \sum_{x \in \mathbb{Z}/p\mathbb{Z}} \exp(-2\pi i \frac{\alpha' x^2}{p}) & \text{if } \alpha = p\alpha' \text{ is a unit.} \end{cases}$$
Therefore, according to the theory of Gauss sums:
$$\widetilde{\gamma}_p(\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ is a unit ,} \\ (\alpha', p)_p & \text{if } \alpha = p\alpha' \text{ is a unit and } p \equiv 1 \ (4), \\ -i(\alpha', p)_p & \text{if } \alpha = p\alpha' \text{ is a unit and } p \equiv 3 \ (4). \end{cases}$$
2. *For* ∞*:*
$$\gamma_\infty(1) = \rho \qquad \gamma_\infty(-1) = \rho^{-1}$$
3. *For* $p = 2$*:*
$$\widetilde{\gamma}_2(\alpha) = \begin{cases} 2^{-\frac{1}{2}} \sum_{x \in \mathbb{Z}/2\mathbb{Z}} \exp(-2\pi i \frac{\alpha x^2}{4}) & \text{if } \alpha \text{ is a unit,} \\ 4^{-\frac{1}{2}} \sum_{x \in \mathbb{Z}/4\mathbb{Z}} \exp(-2\pi i \frac{\alpha' x^2}{8}) & \text{if } \alpha = p\alpha' \text{ is a unit.} \end{cases}$$

α	1	3	5	7	2	6	10	14
$\widetilde{\gamma}_2$	ρ^{-1}	ρ	ρ^{-1}	ρ	ρ^{-1}	ρ^{-3}	ρ^3	ρ

We note also the product formula for a quadratic form Q defined over $\mathbb{Q}$ which can be proven using Poisson summation:

$$\prod_\nu \widetilde{\gamma}_\nu(Q) = 1.$$

From this product formula follows, for example, the law of quadratic reciprocity as well as the statement that a definite unimodular lattice over $\mathbb{Z}$ can only exist if $8|\dim(V)$. Also the sign in the Gauss sum can be obtained by knowing γ_∞ only.
For more explicit formulas for Weil representations see also [3].

5 Relation of parabolic induction with the Weil representation

Let R be one of $\mathbb{Q}_p, \mathbb{R}, \mathbb{A}$.
Let V be a $\mathbb{Q}$-vectorspace with binary quadratic form and discriminant D, and let $\chi_V : x \mapsto (x, D)_R$ the character given by the Hilbert symbol.

Lemma 5.1. *For any (K_∞-finite if R=$\mathbb{R}, \mathbb{A}$) vector $\varphi \in S(V_R)$ (the Schwartz space), the function*

$$\lambda(\varphi) : g \mapsto (\omega(g)\varphi)(0)$$

is in $I_R(\chi_V)$.

Proof. Follows from the explicit formulæ of the Weil representation: We have by (7):

$$\Phi_\varphi(m(a)\, g) = \chi_V(a)\, |a|\, (\omega(g)\varphi)(0) = \chi_V(a)\, |a|\, \Phi_\varphi(g)$$

and by (8):

$$\Phi_\varphi(n(b)g) = \Psi(\frac{b}{2}0^2)\, (\omega(g)\varphi)(0) = \Phi_\varphi(g)$$

□

In other words we get a morphism of $\mathrm{SL}_2(R)$-representations

$$\lambda : S(V_R) \to I_R(\chi_V)$$

Denote the image by $I(V_R)$.
Fix now χ any continuous quadratic character $\mathbb{A}^*/\mathbb{Q}^* \to \{\pm 1\}$.

Theorem 5.2 (Kudla)**.**

$$I_{\mathbb{Q}_\nu}(\chi_\nu) = \bigoplus_{[V_\nu]} I_{\mathbb{Q}_\nu}(V_\nu)$$

where $[V_\nu]$ runs over the isomorphism classes of binary quadratic spaces over $\mathbb{Q}_\nu$ such that $\chi_{V_\nu} = \chi_\nu$ (if p is odd, there are two of them, if χ_ν is non-trivial, and one otherwise, see Appendix B).

Consider the following conditions on a collection of $\{V_\nu\}_\nu$, where all V_ν have the same dimension. Assume that the character of V_ν is given by χ_ν.

(Coh) There is a global quadratic space V such that $V \otimes_{\mathbb{Q}} Q_\nu \cong V_\nu$.

(Coh') There is a global quadratic space V such that $V \otimes_{\mathbb{Q}} Q_\nu \cong V_\nu$ for almost all ν.

(B) Almost all Hasse invariants $\varepsilon_p(V_\nu)$ are 1 and $\prod \varepsilon_p(V_\nu) = 1$

(B') Almost all Hasse invariants $\varepsilon_p(V_\nu)$ are 1.

Theorem 5.3. *We have*

$$\mathrm{B} \Leftrightarrow \mathrm{Coh} \qquad \mathrm{B}' \Leftrightarrow \mathrm{Coh}'$$

We call $\{V_\nu\}_\nu$ a **coherent collection** if B (or equivalently Coh) holds and an **incoherent collection** if B' holds, but $\prod \varepsilon_p(V_\nu) = -1$.

Corollary 5.4. *We have*

$$I_{\mathbb{A}}(\chi) = \bigoplus_{[V]} I_{\mathbb{A}}(V) \oplus \bigoplus_{[\{V_\nu\}_\nu]} I_{\mathbb{A}}(\{V_\nu\}_\nu)$$

where $[V]$ runs over the isomorphism classes of binary quadratic spaces over $\mathbb{Q}$ such that $\chi_V = \chi$ and $\{V_\nu\}_\nu$ runs over incoherent collections of binary quadratic spaces with $\chi_{V_\nu} = \chi_\nu$.

Proof. This follows because the tensor product is the restricted one w.r.t. the functions ξ_p^0. Those lie in the subspace coming from the representation with Hasse invariant 1. □

We will now explicitly determine the image in $I_{\mathbb{Q}_\nu}(\chi_V)$ of some special functions $\varphi \in S(V_{\mathbb{Q}_\nu})$.

Proposition 5.5. *Let $V_{\mathbb{Q}_p}$ be a quadratic vector space and $M \subset V_{\mathbb{Q}_p}$ a* unimodular *$\mathbb{Z}_p$-lattice and let φ_M be the characteristic function of M. Note that the existence of M implies that the character χ_{V_p} is unramified, i.e. $\chi_{V_p}(\mathbb{Z}_p^*) = 1$ and that the Hasse invariant is* 1*. Then we have*

$$\lambda(\varphi_M) = \xi_p^0.$$

Proof. Unraveling the definition, we have to show that

$$(\omega(\begin{pmatrix} \alpha & x \\ & \alpha^{-1} \end{pmatrix} k)\varphi_M)(0) = \chi(\alpha)$$

for $k \in \mathrm{SL}_2(\mathbb{Z}_p)$. By formulas (7–8) this boils down to

$$(\omega(k)\varphi_M)(0) = 1$$

but actually we even have

$$\omega(k)\varphi_M = \varphi_M$$

for $\mathrm{SL}_2(\mathbb{Z}_p)$ is generated by $B(\mathbb{Z}_p)$, under which φ_M is clearly invariant because M is unimodular, and w. The Fourier transform leaves φ_M invariant because of Lemma 4.2. □

5.6. Let $K_0(p)$ be the group introduced in 2.1. χ defines a character $K_0(p) \to S^1$. Consider the subspace of $I(\chi|\cdot|^s)$ of those functions, which are right equivariant w.r.t this character. This space is 2 dimensional, generated by 2 functions which are determined by

$$\xi_p^1(1) = 1 \quad \xi_p^1(w) = 0 \quad \xi_p^w(1) = 0 \quad \xi_p^w(w) = 1$$

Proof. By the Iwasawa decomposition any function in $I(\chi|\cdot|^s)$ is determined by their values on $K_0(p)\backslash \mathrm{SL}_2(\mathbb{Z}_p)$. Being right equivariant means that even the values on the following double cosets are determined by one of them:

$$K_0(p)\backslash \mathrm{SL}_2(\mathbb{Z}_p)/K_0(p) = B(\mathbb{F}_p)\backslash \mathrm{SL}_2(\mathbb{F}_p)/B(\mathbb{F}_p)$$

There are just 2 such cosets, represented by 1 and w. □

Proposition 5.7. *Let $p \equiv 3\ (4)$ be a prime and $V_{\mathbb{Q}_p} = \mathbb{Q}_p^2$ with quadratic form of the form $Q : \epsilon_1 x^2 + \epsilon_2 p y^2$. (This implies that the lattice $M := \mathbb{Z}_p^2$ is maximal integral.) We have $M^\vee = \mathbb{Z}_p \oplus \frac{1}{p}\mathbb{Z}_p$. In this case $\lambda(\varphi_M)$ lies in the subspace of $I(\chi|\cdot|^s)$ of functions which right equivariant w.r.t this character and is equal to*

$$\lambda(\varphi_M) = \xi_p^1 + \widetilde{\gamma}(Q)p^{-\frac{1}{2}}\xi_p^w.$$

We denote these functions by $\Phi_p^\pm$ according to whether $\widetilde{\gamma}(Q) = \pm i$. These are precisely the ones described in the introduction (cf. also section 7).

Proof. We have the decomposition $K_0(p) = N^-(p\mathbb{Z}_p)M(\mathbb{Z}_p)N(\mathbb{Z}_p)$, where N^- is the opposite unipotent. Using the formulas of the Weil representation (7–9) and Lemma 4.2 we get:

$$\begin{aligned}
(\omega(m(a))\varphi_M)(0) &= \chi(a) \qquad \text{for } a \in {\mathbb{Z}_p}^* \\
(\omega(n(b))\varphi_M)(0) &= 1 \\
(\omega(n^-(pc))\varphi_M)(0) &= (\omega(w\, n(pc)\, w^{-1})\varphi_M)(0) = (\omega(w\, n(pc))|M|\widetilde{\gamma}(Q)^{-1}\varphi_{M^\vee})(0) \\
&= (|M^\vee|\,|M|\,\varphi_M)(0) = 1
\end{aligned}$$

It follows that $\lambda(\varphi_M)$ is right equivariant w.r.t. this character. It remains to determine

$$\lambda(\varphi_M)(1) = 1$$

and by equation (9) and Lemma 4.2, we have $\omega(w)\varphi_M = \widetilde{\gamma}(Q)|M|\varphi_{M^\vee}$, hence

$$\lambda(\varphi_M)(w) = \widetilde{\gamma}(Q)p^{-\frac{1}{2}}.$$

□

Proposition 5.8. *Let $V_{\mathbb{R}}$ be a* positive definite *real vector space of dimension* 2. *Then there is a* unique *function φ_∞^1 such that*

$$\lambda(\varphi_\infty^1) = \xi_\infty^1.$$

Proof. ξ^1_∞ is the unique vector (up to scalar) in $I_\infty(\chi_\infty)$, on which K_∞ acts by $k_z \cdot \xi^1_\infty = z \cdot \xi^1_\infty$. Hence we have to look for vectors in $S(V_\mathbb{R})$ which transform the same way. Let

$$H := \begin{pmatrix} & -i \\ i & \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C}).$$

Since $K_\infty = \{k_{e(\lambda)} = e(\lambda H) \mid \lambda \in \mathbb{R}\}$ it is the same to ask for an element φ_∞ with

$$H\varphi_\infty = k\varphi_\infty.$$

Now we have:

$$(\mathrm{d}\,\omega(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})\varphi)(x) = 2\pi i \frac{x^2}{2}\varphi(x)$$

as is easily obtained by deriving equation (8). Using the equation $\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = w^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} w$, we get that $\mathrm{d}\,\omega(\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix})$ is the Fourier transform of the operator $\mathrm{d}\,\omega(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})$, therefore

$$\mathrm{d}\,\omega(\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}) = \frac{1}{4\pi i}\Delta$$

where Δ is the Laplace operator w.r.t. the quadratic form on $V_\mathbb{R}$. Hence

$$H = \pi x^2 - \frac{\Delta}{4\pi}$$

For a positive definite space the differential equation

$$\pi x^2 \varphi(x) - \frac{1}{4\pi}\Delta\varphi(x) = \frac{n}{2}\varphi(x)$$

has a unique solution (up to scalar) namely the **Gaussian**

$$\varphi^1_\infty(x) = e^{-\pi x^2}.$$

Since $\varphi^1_\infty(0) = 1$, we must have indeed $\Phi(\varphi^1_\infty) = \xi^1_\infty$.
Functions of higher weights $3, 5, 7, \ldots$ one obtains for example by applying the weight raising and lowering operators:

$$\frac{1}{2}\,\mathrm{d}\,\omega(\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}) \qquad \frac{1}{2}\,\mathrm{d}\,\omega(\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix})$$

The eigenspaces are, however, not one-dimensional anymore. □

Remark 5.9. *Similarly, we get that the functions Φ^k_∞ for $k = -1, -3, -5, \ldots$ arise from the negative definite space, whereas all even functions Φ^k_∞ for $k = \ldots, -2, 0, 2, \ldots$ arise from the indefinite space. Hence we reobtain the well-known statement that for $\chi_\infty = \mathrm{sign}$, the principal series representation $I_\infty(\chi_\infty)$ decomposes into 2 "limit of discrete series representations", whereas for $\chi_\infty = 1$ it is irreducible.*

6 Fourier expansion of the Eisenstein series

Let a factorizable section of $\Phi = \prod \Phi_\nu$ of $I_{\mathbb{A}}(\chi|\cdot|_{\mathbb{A}}^s)$ as in 3.1 be given, such that its restriction to $K_\infty \operatorname{SL}_2(\widehat{\mathbb{Z}})$ is independent of s. We assume that the standard additive character $\Psi : \mathbb{A} \to S^1$ has been chosen (4.7).
The function

$$x \to E_\Phi(n(x)\, g)$$

is continuous on the compact group $\mathbb{A}/\mathbb{Q}$ and hence has a Fourier expansion. Defining

$$c_t(g; s) := \int_{\mathbb{A}/\mathbb{Q}} E(n(x)\, g)\Psi(-tx)\,\mathrm{d}\, x$$

for $t \in \mathbb{Q}$, we get

$$E_\Phi(g) = \sum_{t\in\mathbb{Q}} c_t(g; s)$$

These Fourier "coefficients" satisfy

$$c_t(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g; s) = \Psi(tx)\, c_t(g; s).$$

Lemma 6.1. *If f is a modular form of weight k with Fourier expansion*

$$f(\tau) = \sum_{t\in\mathbb{Q}} a_t(\nu) e(t\tau)$$

then the Fourier "coefficients" associated with ϕ_f are determined by:

$$c_t(\begin{pmatrix} \nu^{1/2} & \mu\nu^{-1/2} \\ & \nu^{-1/2} \end{pmatrix}_\infty 1_f; s) = \nu^{k/2} a_t(\nu) e(t\tau)$$

where $\tau = \mu + \nu i$.

We now proceed to calculate the Fourier expansion of E:
We write

$$B(\mathbb{Q})\backslash \operatorname{SL}_2(\mathbb{Q}) = \{e\} \cup \{w\, n(b) \mid b \in \mathbb{Q}\}$$

We have hence

$$E_\Phi(g; s) = \Phi(s)(g) + \sum_{b\in\mathbb{Q}} \Phi(s)(w\, n(b)\, g)$$

and hence

$$c_t(g; s) = \int_{\mathbb{A}/\mathbb{Q}} \Phi(s)(n(x)\, g)\Psi(-tx)\,\mathrm{d}\, x + \sum_{b\in\mathbb{Q}} \int_{\mathbb{A}/\mathbb{Q}} \Phi(s)(w\, n(b+x)\, g)\Psi(-tx)\,\mathrm{d}\, x$$

$$= \int_{\mathbb{A}/\mathbb{Q}} \Phi(s)(g)\Psi(-tx)\,\mathrm{d}\, x + \int_{\mathbb{A}} \Phi(s)(w\, n(x)\, g)\Psi(-tx)\,\mathrm{d}\, x$$

Note that the first coefficient is non-zero only if $t = 0$. I.e.

$$c_t(g;s) \quad = \quad \delta_{t,0}\Phi(s)(g) + W_t(\Phi(s), g)$$

where

$$W_t(\Phi(s), g) := \int_{\mathbb{A}} \Phi(s)(w\, n(x)\, g)\Psi(-tx)\, \mathrm{d}\, x$$

If $\Phi(s)$ is factorizable, as above, we have:

$$W_t(\Phi(s), g) = \prod_\nu W_{\nu,t}(\Phi_\nu(s), g)$$

where

$$W_{\nu,t}(\Phi_\nu(s), g) \quad = \quad \int_{\mathbb{Q}_\nu} \Phi_\nu(s)(w\, n(x)\, g)\Psi_\nu(-tx)\, \mathrm{d}\, x$$

is called the **Whittaker integral.**
The functions $W_t(\Phi(s), g)$ and hence the whole Eisenstein series can always be meromorphically continued to all $s \in \mathbb{C}$. We will not discuss this here, although it will follow for the specific Eisenstein series from the explicit calculations. For the local factors, analytic continuation is easier to see, and we will write $W_{\nu,t}(\Phi, g; s)$ for the analytically continued function.

6.2. Let $k = \mathbb{Q}(\sqrt{-q})$ with $q \equiv 3\ (4)$ prime be the imaginary quadratic field of the introduction. We consider the quadratic space $V := k$ with quadratic form given by the norm N. The associated character is $\chi_V(a) = (a, -q)$.
There is one and only one way to turn $\{V_\nu\}_\nu$ into an *incoherent* collection with the same character such that

- There remains a vector $\Phi \in I(\{V_\nu\}_\nu)$ which is χ-equivariant under $K_0(q)$.

Namely we have to change V_q which has Hasse invariant $\varepsilon = 1$ (because N represents 1) into the space V_q^- with the same discriminant and Hasse invariant $\varepsilon = -1$. From the explicit formula 4.10 follows that

$$\widetilde{\gamma}(Q) = -\varepsilon(Q)\, i$$

where Q is the corresponding quadratic form. Therefore by Propositions 5.5–5.8, in both cases there is a function

$$\prod_p \varphi_p \varphi_\infty$$

such that applying Φ, we get the section

$$\Phi^\pm := \Phi_q^\pm \prod_{p \neq q} \Phi_p \Phi_\infty^1$$

according to the case. We will now compute the Fourier coefficients explicitly for the Eisenstein series associated with these two sections.

6.3. For $t \in \mathbb{Z}$ write
$$\rho(t) = \{\mathfrak{a} \subset \mathcal{O}_k \mid N(\mathfrak{a}) = t\}.$$
This is a multiplicative function and indeed we have
$$\rho(t) = \prod_\nu \rho_\nu(t)$$
where
$$\rho_\nu(t) = \begin{cases} \mathrm{ord}_p(t) + 1 & \nu = p \neq q, \chi_p(p) = 1 \text{ (i.e. for } p \text{ split in } k) \\ \frac{1}{2}(1 + \chi_\nu(t)) & \text{otherwise} \end{cases}$$

Proof. We have $\rho_p(t) = \rho(p^{\mathrm{ord}_p(t)})$ except for $p = q$. But if $\rho_q(t) = 0$, i.e. if $\chi_q(t) = -1$ then also $\rho(t) = 0$ by the product formula $\prod_\nu \chi_\nu(t) = 1$. □

6.1 The Siegel-Weil formula

This section is not important for the computation that follows and is included to illustrate Weil's original purpose of introducing the Weil representation. Let V be a positive definite quadratic space over $\mathbb{Q}$, $M_\mathbb{Z}$ a lattice in it, φ_∞ be the Gaussian, and $\varphi_f = \varphi_M$ be the characteristic function of $M = M_\mathbb{Z} \otimes \widehat{\mathbb{Z}}$. We can form the theta function associated with $\varphi := \varphi_\infty \varphi_f$
$$\theta(\varphi, g) = \sum_{v \in V} (\omega(g)\varphi)(v) \tag{11}$$
A simple calculation shows that it corresponds via the correspondence between adelic and classical modular forms to the series
$$\Theta(\tau) = \sum_{v \in M_\mathbb{Z}} e(v^2 \tau)$$
The fact that it is a modular form follows from Poisson summation. We have the Siegel-Weil formula which holds unconditionally if $\dim(V) > 4$:
$$E(\Phi(\varphi), g; \frac{n}{2} - 1) = \frac{1}{\tau(\mathrm{SO}(V))} \int_{\mathrm{SO}(V_\mathbb{A})/\mathrm{SO}(V_\mathbb{Q})} \theta(h\varphi, g) \,\mathrm{d}\, h$$
where $\mathrm{d}\, h$ is the Tamagawa measure.
We want to illustrate the equality further, by computing the Fourier expansion of the right hand side. We have
$$\theta(\varphi, g) = \sum_{t \in \mathbb{Q}} \theta_t(\varphi, g)$$
setting
$$\theta_t(\varphi, g) := \sum_{v \in V, v^2 = t} (\omega(g)\varphi)(v)$$
This is the Fourier expansion of θ. Hence the Fourier coefficient of the RHS equals
$$\int_{\mathrm{SO}(V_\mathbb{A})/\mathrm{SO}(V_\mathbb{Q})} \sum_{v \in V, v^2 = t} (\omega(g)\varphi)(h^{-1}v) \,\mathrm{d}\, h$$

This is equal to

$$\tau(\mathrm{SO}(v^\perp)) \int_{v\in V_{\mathbb{A}}, v^2=t} (\omega(g)\varphi)(v)\,\mathrm{d}\,v$$

Where τ is the volume of $\mathrm{SO}(v^\perp_{\mathbb{A}})/\,\mathrm{SO}(v^\perp_{\mathbb{Q}})$ and $\mathrm{d}\,v$ is a suitable measure on the sphere. The integral also decomposes as the product over all ν of

$$\int_{v\in V_{\mathbb{Q}_\nu}, v^2=t} (\omega(g)\varphi_\nu)(v)\,\mathrm{d}\,v$$

Now we have

Lemma 6.4. *For $t \neq 0$ and if $\dim(V) > 4$:*

$$W_{\nu,t}(\lambda(\varphi), t; 0) = \widetilde{\gamma}_\nu(Q) \int_{v\in V_{\mathbb{Q}_\nu}, v^2=t} (\omega(g)\varphi_\nu)(v)\,\mathrm{d}\,v.$$

This already proves the equality of the non-zero Fourier coefficients.
The Siegel-Weil formula continues to hold beyond the range $\dim(V) > 4$ if the Eisenstein series is analytically continued. If $\dim(V) = 1, 2$ an additional factor 2 is occurring. We will see this in a special case for $\dim(V) = 2$.

6.2 Local computation

The following general Lemma will be used in the unramified and ramified cases alike.

Lemma 6.5. *Let p be any prime. Let χ be a quadratic character of $\mathbb{Q}_p^*$ such that $\chi(1+\mathbb{Z}_p) = 1$ (this is only a restriction for $p = 2$) and let $\Phi_p \in I_{\mathbb{Q}_p}(\chi|\cdot|^s)$ be $K_0(p)$-equivariant character w.r.t χ. Then we have*

$$W_{p,t}(\Phi_p, 1; s) = X_{\mathbb{Z}_p}(t)\,\Phi_p(w) + \left(\sum_{i=1}^{\infty}(\chi(p)p^{-s})^i \int_{\mathbb{Z}_p^*} \chi(b)\Psi_p(btp^{-i})\,\mathrm{d}\,b\right)\Phi(1)$$

Proof. Let us abbreviate W_t for the Whittaker integral and Φ_p for $\Phi_p(s)$, s being fixed. We decompose $\mathbb{Q}_p = \mathbb{Z}_p + \bigcup_{i=1}^{\infty} p^{-i}\mathbb{Z}_p^*$, hence

$$W_t = \int_{\mathbb{Z}_p} \Phi_p(w\,n(b))\Psi_p(-bt)\,\mathrm{d}\,b + \sum_{i=1}^{\infty} \int_{p^{-i}\mathbb{Z}_p^*} \Phi_p(w\,n(b))\Psi_p(-bt)\,\mathrm{d}\,b$$

The first integral is just equal to $\Phi(w)$, if $t \in \mathbb{Z}_p$ and 0 otherwise. On the second, we may use the Iwasawa decomposition

$$m(-b^{-1})\,n(-b)\begin{pmatrix}1 & \\ b^{-1} & 1\end{pmatrix} = w\,n(b).$$

Note that the matrix $\begin{pmatrix}1 & \\ b^{-1} & 1\end{pmatrix}$ is in $K_0(p)$ and χ is 1 on it. Therefore:

$$\Phi_p(w\,n(b)) = \Phi_p(m(-b^{-1})\,n(-b)\begin{pmatrix}1 & \\ -b^{-1} & 1\end{pmatrix})) = \chi(-b)|b|^{-s-1}\,\Phi_p(1).$$

Hence we get:

$$W_t = X_{\mathbb{Z}_p}(t)\Phi_p(w) + \sum_{i=1}^{\infty} \chi(p^i)(p^{-s-1})^i p^i \int_{\mathbb{Z}_p^*} \chi(b)\Psi_p(btp^{-i})\,\mathrm{d}\,b.$$

□

Proposition 6.6. *If Φ_p and χ are unramified, i.e. constant (and equal to 1) on $\mathrm{SL}_2(\mathbb{Z}_p)$, resp. $\mathbb{Z}_p^*$, we have*

$$W_{p,t}(\Phi_p, 1; s) = L_p(\chi, s+1)^{-1} \sum_{i=0}^{\mathrm{ord}(t)} (\chi(p)p^{-s})^i$$

and 0 if $t \notin \mathbb{Z}_p$. In particular

$$W_{p,0}(\Phi_p, 1; s) = L_p(\chi, s+1)^{-1} L_p(\chi, s).$$

Proof. Let us abbreviate W_t for the Whittaker integral. By Lemma 6.5, we have if $t \in \mathbb{Z}_p$:

$$W_t = 1 + \sum_{i=1}^{\infty} \chi(p^i)(p^{-s})^i \int_{\mathbb{Z}_p^*} \Psi_p(btp^{-i})\,\mathrm{d}\,b.$$

Using the Lemma 6.8, we get

$$W_t = \sum_{i=0}^{\mathrm{ord}(t)} (\chi(p)p^{-s})^i - p^{-1} \sum_{i=1}^{\mathrm{ord}(t)+1} (\chi(p)p^{-s})^i = L(\chi, s+1)^{-1} \sum_{i=0}^{\mathrm{ord}(t)} (\chi(p)p^{-s})^i$$

and that $W_t = 0$ if $t \notin \mathbb{Z}_p$. This holds also setting $\mathrm{ord}(t) = \infty$ for $t = 0$. □

Proposition 6.7. *Now assume that $q \equiv 3$ (4) and χ is the ramified quadratic character with $\chi(q) = 1$ and that Φ is $K_0(q)$-equivariant w.r.t. χ. We have then:*

$$W_{q,t}(\Phi_q, 1; s) = \begin{cases} \Phi_q(w) - (q^{-s})^{\mathrm{ord}(t)+1} i q^{-\frac{1}{2}} \chi(t)\, \Phi_q(1) & t \in \mathbb{Z}_p^*, \\ \Phi_q(w) & t = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let us abbreviate W_t for the Whittaker integral. By Lemma 6.5, we have if $t \in \mathbb{Z}_p$:

$$W_t = \Phi_q(w) + \left(\sum_{i=1}^{\infty} \chi_q(q^i)(q^{-s})^i \int_{\mathbb{Z}_q^*} \chi(b)\Psi_q(btq^{-i})\,\mathrm{d}\,b \right) \Phi_q(1).$$

Using Lemma 6.9, we get (since $\chi_q(q) = 1$) if $t \in \mathbb{Z}_p^*$:

$$W_t = \Phi_q(w) + (q^{-s})^{\mathrm{ord}(t)+1} i q^{-\frac{1}{2}} \chi_q(-t)\Phi_q(1)$$

and $W_t = \Phi_q(w)$ if $t = 0$ and $W_t = 0$ otherwise. □

Define

$$W^*_{\nu,t}(\Phi(s);g) := \Lambda_\nu(\chi, s+1)\, W_{\nu,t}(\Phi(s);g)$$

where

$$\Lambda_\nu(\chi, s+1) = \begin{cases} \frac{1}{1-\chi_p(p)p^{-s-1}} & \nu = p \neq q \\ 1 & \nu = q \\ \pi^{-\frac{s}{2}-1}\Gamma(\frac{s}{2}+1) & \nu = \infty \end{cases}$$

is the normalized L-series associated with χ.
Some "easy" Fourier transforms used above:

Lemma 6.8.

$$\int_{\mathbb{Z}_p^*} \Psi(-xt)\,\mathrm{d}\,x = \begin{cases} 0 & 0 > \mathrm{ord}(t)+1 \\ -p^{-1} & 0 = \mathrm{ord}(t)+1 \\ 1-p^{-1} & 0 < \mathrm{ord}(t)+1 \end{cases}$$

Lemma 6.9. *If $q \equiv 3\ (4)$ is a prime and $\chi(a) = (a,q)$ the ramified quadratic character at q with $\chi_q(q) = 1$.*

$$\int_{\mathbb{Z}_q^*} \chi(x)\Psi(-xt)\,\mathrm{d}\,x = \begin{cases} 0 & 0 \neq \mathrm{ord}(t)+1 \\ iq^{-\frac{1}{2}}\chi(t) & 0 = \mathrm{ord}(t)+1 \end{cases}$$

Proposition 6.10. *For $t \neq 0$, we have for $p \neq q$:*

$$W^*_{p,t}(\Phi_p, 1; 0) = \rho_p(t) \qquad W^*_{\nu,t}(\Phi_p, 1; 0) = 0 \ \Leftrightarrow\ \chi_p(t) = -1,$$

and for $\nu = q$

$$W^*_{q,t}(\Phi_q^{\pm}, 1; 0) = \mp 2iq^{-\frac{1}{2}}\rho_q(\pm t) \qquad W^*_{q,t}(\Phi_q^{\pm}, 1; 0) = 0 \ \Leftrightarrow\ \chi_q(t) = \mp 1,$$

and for $\nu = \infty$

$$W^*_{\infty,t}(\Phi_\infty, g_\tau; 0) = 2i\rho_\infty(t)\nu^{\frac{1}{2}}e(t\tau) \qquad W^*_{\infty,t}(\Phi_\infty, 1; 0) = 0 \ \Leftrightarrow\ \chi_\infty(t) = -1.$$

If $W^*_{\nu,t}(\Phi^-_\nu, 1; 0) = 0$ then

$$\frac{\mathrm{d}}{\mathrm{d}\,s} W^*_{\nu,t}(\Phi^-_\nu, g_\tau; 0)|_{s=0} = \begin{cases} \frac{(\mathrm{ord}_p(t)+1)}{2}\log(p) & \nu = p \neq q \\ iq^{-\frac{1}{2}}(\mathrm{ord}_q(t)+1)\log(q) & \nu = q \\ i\nu^{\frac{1}{2}}\mathrm{Ei}(4\pi t\nu)e(t\tau) & \nu = \infty \end{cases}$$

Proof. For $t \neq 0$, $p \neq q$, using Proposition 6.6:

$$W^*_{p,t}(\Phi_p, 1; 0) = \sum_{i=0}^{\mathrm{ord}_p(t)} \chi_p(p)^i = \begin{cases} \mathrm{ord}_p(t)+1 & \chi_p(p) = 1 \\ 1 & \chi_p(p) = -1 \ \ \mathrm{ord}_p(t) \cong 0\ (2) \\ 0 & \chi_p(p) = -1 \ \ \mathrm{ord}_p(t) \cong 1\ (2) \end{cases}$$

Therefore $W^*_{p,t}(\Phi_p, 1; 0) = \rho_p(t) = \rho_p(-t)$. Note that $\chi_p(p) = -1$ and $\mathrm{ord}_p(t) \cong 1$ (2) if and only if $\chi_p(t) = -1$. If it is 0 then

$$\frac{\mathrm{d}}{\mathrm{d}\,s} W^*_{p,t}(\Phi_p, 1; 0) = \frac{(\mathrm{ord}_p(t) + 1)}{2} \log(p)$$

For $t \neq 0$, $p = q$, using Proposition 6.7:

$$W^*_{q,t}(\Phi_q^-, 1; 0) = iq^{-\frac{1}{2}}(1 - \chi(t)) = iq^{-\frac{1}{2}} \begin{cases} 2 & \chi_q(t) = -1 \\ 0 & \chi_q(t) = 1 \end{cases}.$$

$$W^*_{q,t}(\Phi_q^+, 1; 0) = -iq^{-\frac{1}{2}}(1 + \chi(t)) = iq^{-\frac{1}{2}} \begin{cases} 0 & \chi_q(t) = -1 \\ 2 & \chi_q(t) = 1 \end{cases}.$$

This says that $W^*_{q,t}(\Phi_q^\pm, 1; 0) = \mp 2iq^{-\frac{1}{2}}\rho_q(\pm t)$.
If $W^*_{q,t}(\Phi_q^-(0); 1) = 0$ then

$$\frac{\mathrm{d}}{\mathrm{d}\,s} W^*_{q,t}(\Phi_q^-, 1; 0) = iq^{-\frac{1}{2}}(\mathrm{ord}_q(t) + 1)\log(q)$$

For $t \neq 0$, $\nu = \infty$:

$$W^*_{\infty,t}(\Phi_\infty, 1; 0) = 0$$

if and only if $t < 0$.
The case $\nu = \infty$ will be proven in appendix A. □

6.3 Global computation

We retain the explicit situation of paragraph 6.2 and will prove Theorem 1.1 and formula 1. The task of this section is to compute the product

$$W^*_t(\Phi^\pm, g_\tau 1_f; s) := \Lambda(\chi, s+1) W_t(\Phi, g_\tau 1_f; s) = W^*_{\infty,t}(\Phi_\infty, g_\tau; s) \prod_p W^*_{p,t}(\Phi_p^\pm, 1; s)$$

and investigate its properties at $s = 0$.
From the equation

$$w\, n(b)\, m(a) = m(a^{-1})\, w\, n(a^{-2}b)$$

follows that the Whittaker integrals satisfy the property

$$W_{\nu,t}(\Phi(s), n(b)m(a)g) = \chi(a)\, |a|^{s+1}\, \Psi(tb)\, W_{\nu,ta^2}(\Phi(s), g)$$

Hence, in particular:

$$W_{\infty,t}(\Phi(s), g_\tau) = \nu^{\frac{s+1}{2}}\, \Psi(t\mu)\, W_{\infty,t\nu}(\Phi(s), 1)$$

For Φ^+, Proposition 6.10 yields for $t \neq 0$

$$W^*_t(\Phi^+, 1; 0) = \pi^{-1}\rho(t) 4\pi q^{-\frac{1}{2}} e^{-2\pi t}.$$

and hence

$$W_t(\Phi^+, g_\tau\, 1_f; 0)^* = 4\rho(t) q^{-\frac{1}{2}} \nu^{\frac{1}{2}} e(t\tau).$$

Denote by $\mathrm{Diff}(t)$ the set of ν such that $W^*_{\nu,t}(\Phi^-_\nu, g_\tau; 0) = 0$. Because of the product formula

$$1 = \chi_\infty(t) \prod_p \chi_p(t)$$

and Proposition 6.10 for $\Phi = \Phi^-$ an odd number of factors has to vanish at $s = 0$. Hence $W_t(\Phi^-; g_\tau\, 1_f; 0)$ vanishes identically.
If $|\mathrm{Diff}(t)| > 1$, then also the derivative vanishes.

- **Case** $t = 0$**:** We have then for $p \neq q$:

$$\begin{aligned} W^*_{p,0}(\Phi, 1; s) &= L_p(\chi, s) \\ W^*_{q,0}(\Phi^\pm, 1; s) &= \mp i q^{-\frac{1}{2}} L_q(\chi, s) \\ W^*_{\infty,0}(\Phi, g_\tau; s) &= i\nu^{\frac{1-s}{2}} \pi^{-\frac{s+1}{2}} \Gamma(\frac{s+1}{2}) \qquad \text{see [2] for this calculation} \\ W^*_0(\Phi^\pm, g_\tau\, 1_f; s) &= \pm \nu^{\frac{1-s}{2}} q^{-\frac{1}{2}} \Lambda(\chi, s) \end{aligned}$$

  At $s = 0$, we have:

$$c_0^\pm(s) = \Lambda(\chi, s+1)\Phi(0)(g_\tau\, 1_f) + W_0^*(\Phi(0); g_\tau\, 1_f)$$

$$= \Lambda(\chi, s+1)\nu^{\frac{s+1}{2}} \pm \nu^{\frac{1-s}{2}} q^{-\frac{1}{2}} \Lambda(\chi, s)$$

$$= \Lambda(\chi, s+1)\nu^{\frac{s+1}{2}} \pm \nu^{\frac{1-s}{2}} q^{-s} \Lambda(\chi, 1-s)$$

  At 0 we get

$$c_0^+(0) = 2\nu^{\frac{1}{2}} L(\chi, 1) = 2\nu^{\frac{1}{2}} h_k$$

$$c_0^-(0) = 0$$

  (this proves formula (1)) and

$$\frac{\mathrm{d}}{\mathrm{d}\, s} c_0^-(s)|_{s=0} = h_k \left(\log(q) + \log(\nu) + 2\frac{\Lambda'(\chi, 1)}{\Lambda(\chi, 1)} \right)$$

- **Case** $t < 0$**:** Then the derivative is equal to

$$\frac{\mathrm{d}}{\mathrm{d}\, s} W_t^*(\Phi^-, g_\tau\, 1_f; s)|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}\, s} W^*_{\infty,t}(\Phi_\infty, g_\tau; s)|_{s=0} \prod_p W^*_{p,t}(\Phi_p^-, 1; 0)$$

$$= -2q^{-\frac{1}{2}}\, \rho(-t)\, \nu^{\frac{1}{2}}\, \mathrm{Ei}(4\pi t\nu)\, e(t\tau)$$

  Note that $\rho_p(-t) = \rho_p(t)$ for all $p \neq q$.

- **Case** $t>0$, $\mathrm{Diff}(t)=\{p\}$, $p\neq q$**:** We have:

$$\frac{\mathrm{d}}{\mathrm{d}\,s}W_t^*(\Phi^-,g_\tau\,1_f;s)|_{s=0}=W_{\infty,t}^*(\Phi_\infty,g_\tau;s)|_{s=0}\,W_{l,p}^*(\Phi_l^-,1;0)\prod_{l\neq p}W_{l,t}^*(\Phi_l^-,1;0)$$

$$=-\,q^{-\frac{1}{2}}\,(\mathrm{ord}_p(t)+1)\,\log(p)\,\rho(tp^{-1})\,\nu^{\frac{1}{2}}\,e(t\tau)$$

Note that $\mathrm{Diff}(t)=\{p\}$ implies $\mathrm{ord}_t(p)\equiv 1\ (2)$ and $\chi_p(p)=-1$ and hence $\chi_p(p)=(p,-q)_p=-1$ implies $\chi_q(p)=(p,-q)_q=-1$ by quadratic reciprocity. Furthermore $\chi_q(-1)=-1$ since $q\equiv 3\ (4)$. Therefore $W_{q,t}^*(\Phi_q(0);1)=2iq^{-\frac{1}{2}}\rho_q(-t)=2iq^{-\frac{1}{2}}\rho_q(tp^{-1})$.

- **Case** $t>0$, $\mathrm{Diff}(t)=\{q\}$**:** We have:

$$\frac{\mathrm{d}}{\mathrm{d}\,s}W_t^*(\Phi^-,g_\tau\,1_f;s)|_{s=0}=W_{\infty,t}^*(\Phi_\infty,g_\tau;s)|_{s=0}\,W_{l,q}^*(\Phi_l^-,1;s)\prod_{l\neq q}W_{l,t}^*(\Phi_l,1;s)$$

$$=-2\,q^{-\frac{1}{2}}\,(\mathrm{ord}_q(t)+1)\,\log(q)\,\rho(t)\,\nu^{\frac{1}{2}}\,e(t\tau)$$

Note that $\rho_q(-t)=0$ implies $\rho_q(t)=1$.

The last two cases may be summarized by saying:

$$\frac{\mathrm{d}}{\mathrm{d}\,s}W_t^*(\Phi^-,g_\tau\,1_f;0)=-q^{-\frac{1}{2}}\,\nu^{\frac{1}{2}}\,e_p\,(\mathrm{ord}_q(t)+1)\,\log(p)\,\rho(tp^{e_p-2})\,e(t\tau).$$

This finishes the proof of Theorem 1.1 taking into account that for $t\neq 0$:

$$a_t(\nu)e(t\tau)=\nu^{-\frac{1}{2}}q^{\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}\,s}W_t^*(\Phi,g_\tau\,1_f;s)|_{s=0}.$$

7 Comparison with the classical Eisenstein series

7.1. Will will show in this section that the classical analogue of the two Eisenstein series defined adelically are indeed the Eisenstein series of the introduction. We compute:

$$E_\Phi(\begin{pmatrix}\nu^{1/2} & \mu\nu^{-1/2}\\ & \nu^{-1/2}\end{pmatrix}_\infty 1_f;s)\nu^{-1/2}=\nu^{-1/2}\sum_{\gamma\in B(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{Q})}\Phi(s)(\gamma\begin{pmatrix}\nu^{1/2} & \mu\nu^{-1/2}\\ & \nu^{-1/2}\end{pmatrix}_\infty 1_f)$$

w.l.o.g. we may assume that γ is integral.

$$=\nu^{-1/2}\sum_{\gamma\in B(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{Z})}\Phi(s)(\gamma\begin{pmatrix}\nu^{1/2} & \mu\nu^{-1/2}\\ & \nu^{-1/2}\end{pmatrix}_\infty 1_f)$$

We need the Iwasawa decomposition at ∞:

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\nu^{1/2} & \mu\nu^{-1/2}\\ & \nu^{-1/2}\end{pmatrix}=m(\nu^{\frac{1}{2}}|c\tau+d|^{-1})\,n(\cdots)\,k_{\frac{c\tau+d}{|c\tau+d|}}$$

which plugged into ξ^1_∞ yields:

$$\nu^{\frac{1}{2}(s+1)}|c\tau+d|^{s+1}\frac{c\tau+d}{|c\tau+d|}=\nu^{\frac{1}{2}(s+1)}|c\tau+d|^{s}(c\tau+d)$$

For each $p\neq q$, we have

$$\Phi_p(\begin{pmatrix}a & b\\ c & d\end{pmatrix})=1$$

because the matrix is integral. For $p=q$, we distinguish whether $c\equiv 0\ (q)$ or not. If $c\equiv 0\ (q)$, then since Φ_q is $K_0(q)$-equivariant, we get

$$\phi_q(\begin{pmatrix}a & b\\ c & d\end{pmatrix})=\chi_q(a)$$

$c\not\equiv 0\ (q)$, we can write modulo q

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}=n(\frac{a}{c})\,w\,m(-c)\,n(\frac{d}{c})$$

Hence

$$\Phi^{\pm}_q(\begin{pmatrix}a & b\\ c & d\end{pmatrix})=\chi_q(-c)\,\Phi^{\pm}(w)=\pm\chi_q(c)iq^{-\frac{1}{2}}$$

Together this gives the formula of section 1.

A Unconventional calculation of the Whittaker integrals at ∞

Proposition A.1. *For $t\neq 0$, for the function given by $s\in\mathbb{C}$ with $\Re(s)\geq 0$ by*

$$W_{\infty,t}(\Phi_\infty,1;s)=\int_{-\infty}^{\infty}\Phi_\infty(w\,n(b))e(-tb)\,\mathrm{d}\,b$$

we have

$$W_{\infty,t}(\Phi_\infty,1;0)=\begin{cases}2\pi i e^{-2\pi t} & t>0,\\ 0 & t<0,\end{cases}$$

and for $t<0$

$$\frac{\mathrm{d}}{\mathrm{d}\,s}W_{\infty,t}(\Phi(s),1;s)|_{s=0}=\pi i e^{-2\pi t}\mathrm{Ei}(4\pi t).$$

Proof. We have the Iwasawa decomposition:

$$w\,n(b)=m((b^2+1)^{-\frac{1}{2}})\,n(\cdots)\,k_{\frac{-b+i}{b^2+1}}$$

and therefore

$$\Phi_\infty(w\,n(b))=-(b-i)|b^2+1|^{-\frac{s}{2}-1}$$

and

$$W_{\infty,t}(\Phi_\infty, 1; s) = -\int_{-\infty}^{\infty} (b-i)(1+b^2)^{-\frac{s}{2}-1} e(-tb)\,\mathrm{d}\,b.$$

We compute the value at $s = 0$

$$\int_{-\infty}^{\infty} \frac{1}{b+i} e(-tb)\,\mathrm{d}\,b$$

by the residue theorem and get 0 for $t < 0$ and $-2\pi i e^{-2\pi t}$ for $t > 0$ (observe that the winding number at $-i$ is -1).
We will need also the Fourier transform

$$\int_{-\infty}^{\infty} \frac{2b}{1+b^2} e(-tb)\,\mathrm{d}\,b.$$

By the residue theorem we get $2\pi i e^{2\pi t}$ for $t < 0$ and $-2\pi i e^{-2\pi t}$ for $t > 0$ (observe that the winding number at $-i$ is -1).
Now assume that $t < 0$. The derivative at $s = 0$ is

$$-\frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{i+b} \log(1+b^2) e(-tb)\,\mathrm{d}\,b$$

We have seen above that the Fourier transform of $\frac{1}{i+b}$ is $-2\pi i X_{>0}(t)e^{-2\pi t}$ which is a tempered distribution.
Let $D \subset S(\mathbb{R})$ be the subspace of the space of Schwartz functions such that the Fourier transform has support in $\mathbb{R}_{<0}$. Then for $\varphi \in D$, the function $\int_{-\infty} \varphi : \int_{-\infty}^{x} \varphi(t)\,\mathrm{d}\,t$ is well-defined and again in D and $\widehat{\int_{-\infty} \varphi} = \frac{\widehat{\varphi}}{2\pi i x}$.
Therefore for $\varphi \in D$

$$\begin{aligned}
& \langle X_{<0}\frac{e^{2\pi t}}{t}, \widehat{\varphi}\rangle \\
= \quad & \langle 2\pi i \operatorname{sign}(t) e^{-2\pi|t|}, \frac{\widehat{\varphi}}{2\pi i t}\rangle \qquad \text{because } \widehat{\varphi} \text{ has support only in } t<0. \\
= \quad & \langle \frac{2b}{1+b^2}, \widehat{\frac{\widehat{\varphi}}{2\pi i t}}\rangle \\
= \quad & \langle \frac{2b}{1+b^2}, \int_{-\infty} \varphi^-\rangle \\
= \quad & \langle \int_0 \frac{2b}{1+b^2}, \varphi^-\rangle \qquad \text{By the exercise} \\
= \quad & \langle \log(1+b^2), \varphi\rangle \qquad \text{the left hand side is an even function}
\end{aligned}$$

Exercise A.2. *For a Schwartzfunction φ with $\int_{-\infty}^{\infty} \varphi = 0$ and any locally integrable function f which defines a distribution on the Schwartz space, we have:*

$$\langle f, \int_{-\infty} \varphi\rangle = \langle \int_0 f, \varphi\rangle.$$

Now for $\varphi \in D$ also $\varphi\frac{1}{b-i}$ is in D, for the Fourier transform is $-\widehat{\varphi} * 2\pi i X_{<0}e^{2\pi t}$. This convolution preserves support in $\mathbb{R}_{<0}$. Therefore we can compute the Fourier transform of $\log(1+b^2)\frac{1}{i+b}$ as distribution:

$$\begin{aligned}
& \langle \log(1+b^2)\frac{1}{i+b}, \varphi\rangle \\
= & \langle \log(1+b^2), \varphi\frac{1}{b-i}\rangle \\
= & 2\pi i\langle X_{<0}\frac{e^{2\pi t}}{t}, \widehat{\varphi} * X_{<0}e^{2\pi t}\rangle \qquad \text{by the above calculation} \\
= & 2\pi i\langle X_{<0}\frac{e^{2\pi t}}{t} * X_{>0}e^{-2\pi t}, \widehat{\varphi}\rangle
\end{aligned}$$

Furthermore

$$X_{<0}\frac{e^{2\pi t}}{t} * X_{>0}e^{-2\pi t} = e^{-2\pi t}\int_{-\infty}^{t}\frac{e^{4\pi x}}{x}\,\mathrm{d}\,x = e^{-2\pi t}\mathrm{Ei}(4\pi t).$$

□

B Appendix: Quadratic spaces of dimension 2.

Let p an odd prime. A quadratic space over $\mathbb{Q}_p$ of any dimension is determined by its Hasse invariant and discriminant. In dimension 2, if the quadratic form on ${\mathbb{Q}_p}^2$ is given by ax^2+by^2, then the class of $-ab$ modulo squares is its discriminant and $(a,b)_p$ (Hilbert symbol) is its Hasse invariant. If the discriminant is trivial then there is no space with Hasse invariant -1. Otherwise all possibilities occur and are given explicitly by the following quadratic forms where ε represents a non-square in $\mathbb{Q}_p^*$

One the left hand side, we have $p \equiv 1$ (4) (i.e -1 is a square) and on the right hand side $p \equiv 3$ (4).

disc	*Hasse*	a	b	γ
1	1	1	1	1
ε	1	ε	1	1
ε	-1	$p\varepsilon$	p	-1
p	1	p	1	1
p	-1	$p\varepsilon$	ε	-1
εp	1	$p\varepsilon$	1	-1
εp	-1	p	ε	1

disc	*Hasse*	a	b	γ
1	1	1	ε	1
ε	1	1	1	1
ε	-1	p	p	-1
p	1	$p\varepsilon$	1	i
p	-1	p	ε	$-i$
εp	1	p	1	$-i$
εp	-1	$p\varepsilon$	ε	i

a and b are not unique (as classes modulo squares), above we gave examples. There are usually 2 possibilities except for the discriminant one form, which has 4 possibilities. The associated quadratic character is given by

$$\chi_p(x) = (x, -ab)_p$$

For discriminant 1 it is trivial. For discriminant ε it is trivial on $\mathbb{Z}_p^*$ and has $\chi_p(p) = -1$ For discriminant p, it is given on $\mathbb{Z}_p^*$ by the projection to $\mathbb{F}_p^*$ and its unique non-trivial quadratic character and by $\chi_p(p) = 1$. For discriminant $p\varepsilon$ it is the product of these 2.

γ is computed w.r.t. the standard additive character $\Psi_p : \mathbb{Q}_p \to S^1$.

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