Picard stacks and Jacobian stacks of curves

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1 Stacks and Gerbes

2 Picard Stacks

3 The case of curves

Motivation

Let k be a ground field. By the Yoneda Lemma a variety (or scheme) X over k is uniquely determined by its functor of points

$$\begin{aligned} h_X : \mathrm{Sch}_k &\to \mathrm{Sets} \\ T &\to \mathrm{Hom}_k(T, X) \end{aligned}$$

Given a set valued functor F on varieties given by a moduli problem, one says that F is **representable**, if it is isomorphic to a h_X.

Motivation

Often a moduli problem is not representable because of the existence of non-trivial automorphisms. To take them into account, one considers functors

$$h_X : \mathrm{Sch}_k \quad \rightarrow \quad \mathrm{Groupoids}$$

Examples :

1. X a scheme over k

$$\mathcal{PIC}_k(X) : T \mapsto \left\{ \begin{array}{ll} \text{objects:} & \text{line bundles on } X \times_k T \\ \text{morphisms:} & \text{isomorphisms} \end{array} \right\}$$

2. To a scheme X, one associates the same functor h_X as before, using the obvious inclusion

$$\mathrm{Sets} \hookrightarrow \mathrm{Groupoids}\,.$$

Questions

3. G an algebraic group

$$BG : T \mapsto \left\{ \begin{array}{ll} \text{objects:} & G\text{-prinicipal bundles on } T \\ \text{morphisms:} & \text{isomorphisms} \end{array} \right\}$$

Note: $\mathcal{PIC}_k(X)(T) \cong B\mathbb{G}_m(T \times_k C)$.

Questions:

- The functors h_X are **sheaves** (for the étale topology, say). What is the analogous condition for functors with values in groupoids?
- When should a functor with non-trivial automorphisms be called representable?

Sheaves

Definition

A functor

$$F : \mathrm{Sch}_k \to \mathrm{Sets}$$

is called a *sheaf*, if for all coverings[1] $\{U_i \to X\}$ and elements $x_i \in F(U_i)$ such that

$$x_i = x_j \qquad \text{on } U_i \times_X U_j$$

there is a *unique* $x \in F(X)$ giving rise to the x_i.

Fact: h_X is a sheaf.

[1] always referring to the étale topology in these slides

Stacks

Definition

A functor

$$F : \mathrm{Sch}_k \to \mathrm{Groupoids}$$

is called a *stack*, if for all coverings $\{U_i \to X\}$ and objects $x_i \in F(U_i)$ and isomorphisms

$$\varphi_{ij} : x_i \to x_j \qquad \text{on } U_i \times_X U_j$$

such that

$$\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik} \qquad \text{on } U_i \times_X U_j \times_X U_k$$

there is a *unique (up to unique isomorphism)* $x \in F(X)$ giving rise to the x_i.

All examples given before are stacks.

Stacks

There exists a **stackification** analogous to sheafification.

Definition

A stack

$$F : \mathrm{Sch}_k \to \mathrm{Groupoids}$$

is called **representable** (or an **algebraic stack**) if there exists a (nice...) groupoid object in schemes[2]

$$\circlearrowleft M \rightrightarrows O$$

such that F is the stackification of

$$T \mapsto \left\{ \begin{array}{ll} \text{objects:} & \mathrm{Hom}_k(T, O) \\ \text{morphisms:} & \mathrm{Hom}_k(T, M) \end{array} \right\}$$

[2]better: algebraic spaces

Stacks

For example BG is represented by

$$\circlearrowleft G \rightrightarrows \cdot$$

(where $\cdot = \mathrm{spec}(k)$)

Quotient stack

Let G be an algebraic group acting on X then one defines the quotient $[X/G]$ as the stack represented by

$$\circlearrowleft G \times X \underset{\mathrm{pr}_X}{\overset{\text{action}}{\rightrightarrows}} X$$

One can show:

$$[X/G](T) \cong \left\{ \begin{array}{ll} \text{objects:} & G \text{ bundles } B \text{ on } T + \varphi : B \to X \text{ equivariant} \\ \text{morphisms:} & \text{isomorphisms compatible with the } \varphi \end{array} \right\}$$

Fiber products of stacks

Let

$$\begin{array}{ccc} X & & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & Z & \end{array}$$

be a diagram of stacks. One defines the fiber product by

$$(X \times_Y Z)(T) = \left\{ \begin{array}{ll} \text{objects:} & (x, y) \in (X \times Y)(T) + \varphi : p(x) \to q(y) \\ \text{morphisms:} & \text{isomorphisms compatible with } \varphi \end{array} \right\}$$

This is again a stack. It is representable, if X, Y and Z are.

Fiber products (Example)

Example:

$$T \longrightarrow BG = [\cdot/G] \longleftarrow \cdot$$

where T is a scheme and $\cdot = \mathrm{spec}(k)$.
Note that $T \to BG$ classifies a G principal bundle $\mathcal{B} \to T$. Then one has

$$T \times_{BG} \cdot \cong \mathcal{B}$$

In other words

$$\cdot \to BG$$

is the *universal* G principal bundle.

Gerbes

Definition: Gerbe

Let X be a scheme. Algebraic stacks F over X with the property that any two objects $x, y \in F(T)$ over some $f : T \to X$ are locally isomorphic on T are called **gerbes**.

Main example: $BG = [\cdot/G]$ is a gerbe over $\cdot = \mathrm{spec}(k)$.

Lemma

For a gerbe F over X the following are equivalent

- $F(T)$ is non-empty for any $F : T \to X$.
- $F \cong [X/G]$ for an algebraic group G over X (i.e. with trivial action on X).

Gerbes

For each covering $\{U_i \to X\}$ such that $F(U_i) \neq \emptyset$ we get a collection of algebraic groups $G_i := \mathrm{Aut}(x_i)$ over U_i together with isomorphisms

$$\varphi_{ij} : G_i \to G_j \qquad \text{on } U_i \times_X U_j$$

which however satisfy the cocycle condition only *up to conjugation*. This datum is called the **band** of the gerbe. In case that the G_i are Abelian, we can glue a group scheme G over X and call it the band of F and speak of G-gerbes.

Gerbes

Theorem

Let X be a scheme and G be an (Abelian) group scheme.
Equivalence classes of G-gerbes $F \to X$ are in bijection with

$$H^2(X, G).$$

Picard stacks

Turning a scheme into an (Abelian) group scheme is equivalent to making its functor h_X (Abelian) group valued. What is the generalization to groupoid valued functors?

Example

On the groupoid of line bundles $\mathcal{PIC}_k(X)$ we have a functor

$$\otimes : \mathcal{PIC}_k(X) \times \mathcal{PIC}_k(X) \to \mathcal{PIC}_k(X)$$

a neutral object 1, and a functor

$$(-)^{\otimes -1} : \mathcal{PIC}_k(X) \to \mathcal{PIC}_k(X)$$

behaving like an Abelian group structure *up to isomorphism*.

Picard stacks

It's not so easy to make this precise: For example there should be an isomorphism of functors

$$- \otimes (- \otimes -) \;\to\; (- \otimes -) \otimes -$$

such that

$$\begin{array}{ccccc}
(x \otimes (y \otimes z)) \otimes w & & \longrightarrow & & ((x \otimes y) \otimes z) \otimes w \\
\uparrow & & & & \uparrow \\
x \otimes ((y \otimes z) \otimes w) & & & & (x \otimes y) \otimes (z \otimes w) \\
& \nwarrow & & \nearrow & \\
& & x \otimes (y \otimes (z \otimes w)) & &
\end{array}$$

commutes for all objects x, y, z, w.
($(\mathcal{PIC}_k(X), \otimes, 1)$ is a symmetric monoidal category.)

Picard stacks

Definition

A groupoid G with $\otimes, 1, (-)^{\otimes -1}$ as before, 'behaving like an Abelian group structure *up to isomorphism*' is called a **Picard groupoid**.

Slogan: categorification of the notion of Abelian group.

Definition

A stack

$$F : \mathrm{Sch}_k \to \text{Picard groupoids}$$

is called a Picard stack.

Like for group schemes, $\otimes$ and $(-)^{\otimes -1}$ are actually morphisms of stacks.

Deligne's equivalence

1. Consider the (bounded) derived category $\mathcal{D}_k$ of sheaves of Abelian groups on schemes over k. Denote by $\mathcal{D}_k^{[-1,0]}$ the full subcategory of those objects represented by complexes concentrated in degree -1 and 0.
2. Consider the category $\mathcal{P}$ with objects Picards stacks and morphisms being isomorphism classes of morphisms between Picard stacks.

Theorem (Deligne)

There is an equivalence of categories

$$\begin{array}{ccc} \mathcal{D}_k^{[-1,0]} & \cong & \mathcal{P} \\ (C_{-1} \to C_0) & \mapsto & [C_0/C_{-1}] \end{array}$$

Deligne's equivalence

Examples:

1. If G is an Abelian group scheme, $BG = [\cdot/G]$ is Picard and corresponds to the complex

$$(G \to 0)$$

2. For a scheme X, consider the morphism $\pi : X \to \mathrm{spec}(k)$. Then

$$\tau_{\leq 1} R\pi_* \mathbb{G}_{m,X}[1]$$

corresponds to $\mathcal{PIC}_k(X)$.

Corollary

For each Picard stack F one has an exact sequence ($\Leftrightarrow$: part of a distinguished triangle)

$$[\cdot/F_{-1}] \to F \to F_0$$

If F is representable, F is thus a F_{-1}-gerbe on F_0.

Equivalence classes of extensions like this are in bijection with

$$\mathrm{Ext}^2(F_0, F_{-1}).$$

Let X and G be Abelian group schemes (e.g. X is an Abelian variety and $G = \mathbb{G}_m$). Not all G-gerbes on X define a Picard stack F sitting in a sequence

$$[\cdot/G] \to F \to X.$$

(This is analogous to the fact that not all G principal bundles on X have the structure of a group scheme.)
They have to be equivariant w.r.t. the group structure on X. In other words, the gerbe F needs to satisfy that

$$(m^* F) \otimes (\mathrm{pr}_1^* F)^{\otimes -1} \otimes (\mathrm{pr}_2^* F)^{\otimes -1}$$

are trivial on $X \times X$ [3]. Call such G-gerbes **primitive**. They constitute a subgroup $H^2_{\mathrm{prim}}(X, G)$.
We get a homomorphism

$$\mathrm{Ext}^2(X, G) \to H^2_{\mathrm{prim}}(X, G)$$

which is still not an isomorphism, but closer. We will see an example in the case of the Picard stack later.

[3] compare: $\mathcal{L}$ such that $\lambda_{\mathcal{L}}$ is zero...

Cohomology with values in a Picard stack

Using Deligne's equivalence can define

$$H^i(k, P) = \mathbb{H}^i(k, P_{-1} \to P_0) \qquad \text{(Hypercohomology)}$$

From an exact sequence of Picard stacks get

$$\begin{aligned} 0 \to H^{-1}(k, A) \to H^{-1}(k, B) \to H^{-1}(k, C) \\ \to H^0(k, A) \to H^0(k, B) \to H^0(k, C) \to H^1(k, A) \to \cdots \end{aligned}$$

Cohomology with values in a Picard stack

Examples:

1. We always have (Exercise)

$$\begin{aligned} H^{-1}(k, P) &= \mathrm{Aut}(1) \quad \text{where } 1 \in P(k) \text{ is a neutral element} \\ H^0(k, P) &= \{\text{group of isomorphism classes of } P(k)\} \end{aligned}$$

2. We have

$$H^i(k, BG) = H^{i+1}(k, G).$$

3. For a curve X, we have:

$$\begin{aligned} H^{-1}(k, \mathcal{PIC}_k(X)) &= H^0(X, \mathbb{G}_m) \\ H^0(k, \mathcal{PIC}_k(X)) &= H^1(X, \mathbb{G}_m) \\ H^1(k, \mathcal{PIC}_k(X)) &= H^2(X, \mathbb{G}_m) = \mathrm{Br}(X) \end{aligned}$$

(third line follows from $H^2(\overline{X}, \mathbb{G}_m) = 1$, i.e. $R^2\pi_*\mathbb{G}_m = 1$).

Cohomology with values in a Picard stack

Applying this to the canonical exact sequence

$$[\cdot/F_{-1}] \to F \to F_0$$

we get an exact sequence

$$F(k)/\sim \longrightarrow F_0(k) \xrightarrow{\delta} H^2(k, F_{-1})$$

This can be described as follows: Let $x \in F_0(k)$. The preimage $\{x\} \times_{F_0} F$ of x in F is a F_{-1}-gerbe on spec(k) so classified by an element

$$\delta(x) \in H^2(k, F_{-1})$$

The Picard stack for curves and the Jacobian

Theorem

For a curve X over k the Picard stack $\mathcal{PIC}(X)$ is representable and we have an exact sequence

$$B\mathbb{G}_m \to \mathcal{PIC}_k(X) \to \mathrm{Pic}_k(X)$$

where $\mathrm{Pic}_k(X)$ is the quotient or, because it is a scheme, also the course moduli scheme of $\mathcal{PIC}_k(X)$.
The sequence splits, i.e.

$$\mathcal{PIC}_k(X) = B\mathbb{G}_m \times \mathrm{Pic}_k(X)$$

if X has a k-rational point.

We have a exact sequence (everything defined over k)

$$0 \to J(X) \to \mathrm{Pic}_k(X) \to \mathbb{Z} \to 0$$

where $J(X)$ is an Abelian variety. We call it the **Jacobian** of X.

Cohomology with values in the Picard stack

We get the long exact sequence

$$\begin{aligned} 0 = H^1(k, \mathbb{G}_m) \to H^0(k, \mathcal{PIC}_k(X)) \to H^0(k, \mathrm{Pic}_k(X)) \\ \to H^2(k, \mathbb{G}_m) \to H^1(k, \mathcal{PIC}_k(X)) \to H^1(k, \mathrm{Pic}_k(X)) \end{aligned}$$

which yields

$$0 \to \mathcal{PIC}_k(X)(k)/\sim \to J(k) \oplus n\mathbb{Z} \to \mathrm{Br}(k) \to \mathrm{Br}(X) \to H^1(k, J)$$

$n\mathbb{Z} = \ker(\mathbb{Z} \to H^1(k, J))$

$g = 0$

Forms of $\mathbb{P}^1$ over k are parametrized by (the non-Abelian) $H^1(k, \mathrm{PGL}_2)$. (Note that $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$.) From the sequence

$$1 \to \mathbb{G}_m \to \mathrm{GL}_2 \to \mathrm{PGL}_2 \to 1$$

we get an injective boundary homomorphism

$$H^1(k, \mathrm{PGL}_2) \hookrightarrow H^2(k, \mathbb{G}_m) = \mathrm{Br}(k)$$

One can show that if k is a global or local field then the image consists of the elements of order 2.
Let k be a local or global field, $\alpha \in \mathrm{Br}(k)$ an element of order 2. It corresponds to a quaternion algebra over k. Let C_α be the corresponding curve. In this case we get

$$1 \to \mathcal{PIC}(C_\alpha)(k)/\sim \to \mathbb{Z} \to \mathrm{Br}(k) \to \mathrm{Br}(C_\alpha) \to 1$$

The map $\mathbb{Z} \to \mathrm{Br}(k)$ maps 1 to α (Exercise). We therefore get

$$\mathcal{PIC}_k(C_\alpha) = \{\cdots \cup B\mathbb{G}_m \cup (B\mathbb{G}_m)_\alpha \cup B\mathbb{G}_m \cup (B\mathbb{G}_m)_\alpha \cup B\mathbb{G}_m \cup \cdots\}$$

where $(B\mathbb{G}_m)_\alpha$ is the form of $B\mathbb{G}_m$ defined by α.

$g = 1$

Let E be an elliptic curve over k. Forms of E over k are again parametrized by (the non-Abelian) $H^1(k, \mathrm{Aut}(E))$. Now $\mathrm{Aut}(E) = E \rtimes \mathrm{Aut}(E, +)$. We consider a form E_α of E defined by an $\alpha \in H^1(k, E)$. This is also an E principal bundle over k.
If α is non-trivial, we have $E_\alpha(k) = \emptyset$.
We have

$$\mathrm{Pic}_k(E) = \{\cdots \cup E_{-2\alpha} \cup E_{-\alpha} \cup E \cup E_\alpha \cup E_{2\alpha} \cup \cdots\}$$

and thus

$$\mathrm{Pic}_k(E)(k) = E(k) \oplus n\mathbb{Z} \qquad (n = \mathrm{ord}(\alpha))$$

and get

$$1 \to \mathcal{PIC}_k(E_\alpha)(k)/\sim \to \underbrace{H^0(k, \mathrm{Pic}_k(E_\alpha))}_{E(k)\oplus n\mathbb{Z}} \to \mathrm{Br}(k) \to \mathrm{Br}(E_\alpha) \to H^1(k, \mathrm{Pic}_k(E_\alpha)$$

$g = 1$

In other words, we defined a pairing

$$H^1(k, E) \times E(k) \to \mathrm{Br}(k)$$

Theorem (Tate, Lichtenbaum)

If k is p-adic (i.e. $\mathrm{Br}(k) = \mathbb{Q}/\mathbb{Z}$) then this pairing is a perfect pairing, i.e. the two groups are each others topological dual.

The theorem holds also for curves of genus ≥ 2 but the construction of the pairing is not so geometric (?).
Q: What is the resulting map $n\mathbb{Z} \to \mathrm{Br}(k)$? Is it always 0?

$g = 1, k = \mathbb{R}$

We have either

$$E(\mathbb{R}) = \begin{cases} S_1 \\ S_1 \times \mathbb{Z}/2\mathbb{Z} \end{cases} \quad \text{or}$$

and one can show (Exercise) that

$$H^1(\mathbb{R}, E) = \begin{cases} 1 \\ \mathbb{Z}/2\mathbb{Z} \end{cases}$$

We have also (obviously)

$$\mathrm{Hom}(\underbrace{\mathrm{Pic}^0_{\mathbb{R}}(E)(\mathbb{R})}_{E(\mathbb{R})}, \frac{1}{2}\mathbb{Z}/\mathbb{Z}) \cong \begin{cases} 1 \\ \mathbb{Z}/2\mathbb{Z} \end{cases}$$

Remark

We see that for an elliptic curve

$$\mathrm{Ext}^2(E, \mathbb{G}_m) \cong H^2_{\mathrm{prim}}(E, \mathbb{G}_m) \cong H^1(k, E)$$

Theorem (Breen)

In general for an Abelian variety one has an exact sequence

$$0 \to NS(A)(k)/\mathcal{PIC}(A)(k) \to \mathrm{Ext}^2(A, \mathbb{G}_m) \to H^2_{\mathrm{prim}}(A, \mathbb{G}_m) \to H \to 0$$

where H is 2-torsion.

And if k is local or global of characteristic 0 then

$$H^2_{\mathrm{prim}}(A, \mathbb{G}_m) = H^1(k, A)$$

(?)

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