# Fritz Hörmann — MATH 571: Higher Algebra II — Winter 2011

## Exercise sheet 4

**Choose 4 of the 6 exercises!**

1. **A simple Grassmannian.** Let $F$ be a field over characteristic 0. Prove that $x \in \Lambda^2(F^4)$ is a pure tensor, i.e. $x = v \wedge w$, if and only if $x \wedge x = 0$ in $\Lambda^4(F^4)$. Write down an explicit equation for the condition $x \wedge x = 0$. This is called the Plücker relation. Explain that the set

   $$\{< x > \in \mathbb{P}(\Lambda^2(F^4)) \mid x \wedge x = 0\}$$

   parametrizes planes in 4-space $F^4$ (or equivalently "lines" in projective 3-space $\mathbb{P}(F^4)$).

   By $\mathbb{P}(V)$ we mean the projectivization of a vector space $V$. It means that we consider non-zero vectors up to multiplication with a scalar in $F^*$. Equivalently it is the set of lines (1-dimensional subspaces) in $V$.

2. **Multilinear algebra.** Let $F$ be a field and $M$ a vector space over $F$. Prove that

   $$\Lambda^n(M^*) \cong \Lambda^n(M)^*.$$

   Let $R$ be a commutative ring with 2 invertible. Let $M$ be an $R$-module. Prove that

   $$T^2(M) \cong S^2(M) \oplus \Lambda^2(M).$$

   Let $F$ be a field with $n$ invertible. Let $M$ be a vector space over $F$. Prove that

   $$S^n(M)^* \cong S^n(M^*).$$

3. Hom **and** $\otimes$**.** Let $R$ be a commutative ring (for the next 5 exercises). Let $S$ a commutative $R$-algebra. Prove that there is a homomorphism of $S$-modules:

   $$S \otimes_R \mathrm{Hom}_R(M_1, M_2) \to \mathrm{Hom}_S(M_1 \otimes_R S, M_2 \otimes_R S).$$

   Find a counterexample showing that it does not need to be an isomorphism. Prove that it is an isomorphism if $S \otimes_R \cdot$ is an exact functor (we say that $S$ is flat as an $R$-module) and $M_1$ is finitely presented, i.e. there is an exact sequence

   $$R^n \to R^m \to M_1 \to 0.$$

4. **Being zero is a local property.** Prove that the following are equivalent:

   (a) $M_\wp = 0$ for all $\wp \in \mathrm{spec}(R)$,

   (b) $M_\mathfrak{m} = 0$ for all $\mathfrak{m} \in \mathrm{specm}(R)$,

   (c) $M = 0$.

   Here $M_\wp$ is the localization of $M$ by the multiplicative closed subset $R \setminus \wp$. I.e. elements are symbols $\frac{x}{q}$, where $x \in M$ and $q \in R \setminus \wp$ satisfying the usual relations. It is a module over $R_\wp$. Prove that $M_\wp \cong M \otimes_R R_\wp$.

5. **Localizations are flat.** Prove that $M \mapsto M_\wp$ is an exact functor.

6. **Tensor product preserves projectivity.** Let $M_1$ and $M_2$ be projective $R$-modules. Prove that $M_1 \otimes_R M_2$ is a projective $R$-module.

*Please hand in your solutions on Monday, February 14, 2011 in the lecture room*