Fritz Hörmann — MATH 571: Higher Algebra II — Winter 2011
Exercise sheet 3

1. **Yoneda's lemma.** Let $\mathcal{C}$ be any category. Prove Yoneda's lemma: The functor

$$\begin{array}{rcl} \mathcal{C}^{op} & \to & \mathbf{Funct}(\mathcal{C}, \mathbf{Sets}) \\ O & \mapsto & \mathrm{Hom}(O, \cdot) \end{array}$$

   is fully faithful.

2. **A simple universal object.** Consider the category $\mathcal{C}$ whose objects are pairs (S, e) where S is a set and e is a map $S \to S$. Morphisms $(S, e) \to (S', e')$ are maps $\alpha : S \to S'$ such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{\alpha} & S' \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e'} \\ S & \xrightarrow[\alpha]{} & S' \end{array}$$

   is commutative. We have a forgetful functor $V : \mathcal{C} \to \mathbf{Sets}$ given by $(S, e) \mapsto S$.

   (a) Prove by explicit construction that V has a left adjoint F.

   (b) Prove that $F(\{\cdot\})$ (together with its canonical element $\xi \in V(F(\{\cdot\})))$ represents V (here $\{\cdot\}$ denotes a set with 1 element). *Note that this is also equivalent to* $(F(\{\cdot\}), \xi)$ *being an initial object in the category* $\mathcal{C}_V$ *constructed in the lecture.*

   (c) Explain *(in words, not too scrupulously)* that if we denote $(\mathbb{N}, \nu) := F(\{\cdot\})$ and $1 := \xi$ then the representability amounts to the Peano axioms for $\mathbb{N}$ as set of natural numbers, ν as its "successor function" and $1 \in \mathbb{N}$ as the "1" element.

3. **Adjoints.** Let $V : \mathbf{TopSp} \to \mathbf{Sets}$ be the forgetful functor. Show that it has a right and a left adjoint.

4. **Examples of a non-noetherian rings.** Let R be the ring of continuous functions on the interval $[0, 1]$. Prove that R is not noetherian.

   Let R be the subring of $k[X, Y]$ generated by $Y, XY^2, X^2Y^3, X^3Y^4, \ldots, X^iY^{i+1}, \ldots$. Prove that R is not noetherian. *Note that this is a subring of a noetherian ring!*

5. **Artinian rings.** Let R be a noetherian ring. Prove that the following are equivalent:

   (a) R is artinian,

   (b) the topology on $\mathrm{spec}(R)$ is discrete.

   Hint: (a) $\Rightarrow$ (b) was shown in the lecture (even without assuming R noetherian a priori). For the converse, use the noetherian hyphotesis to see that $\mathrm{spec}(R)$ *is a finite set. Use the decomposition theorem 1.3.4. to decompose R into a product of rings. For each constituent (which is then local) you have to show that* $\mathfrak{m}^n = (0)$ *for the maximal ideal* $\mathfrak{m}$ *and some n.*

Please hand in your solutions on Monday, February 7, 2011 in the lecture room