Fritz Hörmann — MATH 571: Higher Algebra II — Winter 2011

Exercise sheet 1

1. **The spectrum of a ring.** Let R be a commutative ring (with 1 as always). Define $\operatorname{spec}(R)$ (respectively $\operatorname{specm}(R)$) as the set of prime (respectively maximal) ideals of R.

   (a) Prove that, for a ring-homomorphism $\varphi : R_1 \to R_2$ there is a map $\operatorname{spec}(\varphi) : \operatorname{spec}(R_2) \to \operatorname{spec}(R_1)$, given by $I \mapsto \varphi^{-1}(I)$.

   (b) Give a counterexample explaining why this map, in general, does not restrict to a map of specm's.

   (c) Define a topology on $\operatorname{spec}(R)$ (resp. $\operatorname{specm}(R)$) by declaring *closed sets* to be the subsets

   $$V(I) := \{p \in \operatorname{spec}(R) \text{ (resp. } \in \operatorname{specm}(R)) \mid I \subseteq p\}$$

   for any ideal $I \subseteq R$. Prove that this is a topology! It is called the Zariski topology.

   (d) Prove that $\operatorname{specm}(R)$ is the set of closed points in $\operatorname{spec}(R)$.

   (e) Prove that "spec" is a contravariant functor from the category of commutative rings to the category of topological spaces.

   (f) For $f \in R$, define $U(f) := \{p \in \operatorname{spec}(R) \mid f \notin p\}$. Prove that these sets are open and define a basis for the Zariski topology.

   (g) Let R_f be the ring R, localized at the multiplicative set $1, f, f^2, \dots$. Denote by $\iota : R \to R_f$ the corresponding homomorphism. Prove that $\operatorname{spec}(\iota) : \operatorname{spec}(R_f) \to \operatorname{spec}(R)$ is injective with image $U(f)$.

   (h) For a prime ideal $p \in \operatorname{spec}(R)$, we call the quotient field $\operatorname{Quot}(R/p)$ the *residue field* at p. Consider each $f \in R$ as a function

   $$f : \operatorname{spec}(R) \to \bigoplus_{q \in \operatorname{spec}(R)} \operatorname{Quot}(R/q)$$

   by letting $f(p)$ be the residue of f mod p in the $q = p$ summand and 0 in all other summands. Determine the set of zeros of f. When does f vanish identically on $\operatorname{spec}(R)$?

2. **Some spectra.** Describe $\operatorname{spec}(\mathbb{Z})$ and $\operatorname{spec}(\mathbb{Z}[X])$ and their topology. Which residue fields occur? How many points are there with given *finite* residue field?

3. **Integral closure.** Consider the category of commutative ring extensions, where objects are homomorphisms $\varphi : R_1 \to R_2$ and morphisms are commutative diagrams

   $$\begin{array}{ccc} R_1 & \xrightarrow{\varphi} & R_2 \\ {\scriptstyle\alpha_1}\downarrow & & \downarrow{\scriptstyle\alpha_2} \\ S_1 & \xrightarrow[\psi]{} & S_2 \end{array}$$

   Prove that the association

   $$[\varphi : R_1 \to R_2] \mapsto [\widetilde{\varphi} : R_1 \to \operatorname{Int}_{R_1}(R_2)]$$

   defines an endofunctor of this category. Here $\widetilde{\varphi}$ is the restriction of φ.

Please hand in your solutions on Monday, January 17, 2011 in the lecture room