Divisorial rings and Cox rings

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In this manuscript $\mathbb{N}$ will always denote natural numbers including 0.

Contents

1 Preliminaries on monoids

Definition 1.1. *An* **Abelian monoid** *is a set with a binary, associative, and commutative operation which has a neutral element. It will often be called just a monoid in this manuscript because we will not deal with non-commutative monoids. A monoid M is called*

- **finitely generated** *if there is a finite set of generators, or equivalently if there is a surjection of monoids $\mathbb{N}^r \twoheadrightarrow M$ for some r.*
- **integral***, if $a + z = b + z$ implies $a = b$,*
- **fine***, if it is finitely generated and integral.*
- **saturated***, if for all $x \in < M >$ (see definition below) with $nx \in M$ for some $n \in \mathbb{N}_{>0}$ is follows that $x \in M$.*

Abelian monoids form a category, denoted by [***Ab mon***].

1.2. We have the adjunctions

$$[\ \mathbb{Q}\textbf{-vs}\ ] \underset{A \mapsto A \otimes_{\mathbb{Z}} \mathbb{Q}}{\overset{\text{forget}}{\rightleftarrows}} [\ \textbf{Ab}\ ] \underset{M \mapsto <M>}{\overset{\text{forget}}{\rightleftarrows}} [\ \textbf{Ab mon}\ ].$$

Here

$$< M > = \bigoplus_{m \in M} \mathbb{Z}[m]$$

modulo the relations $[0] = 0$ and $[m + n] = [m] + [n]$ for all $m, n \in M$. We have the following facts:

Proposition 1.3. *1. $M \hookrightarrow < M >$ iff M is integral. In this case $< M >$ may be defined as the group of differences, i.e. the set (m,n) of pairs in M modulo the (now transitive) relation*

$$(m_1, n_1) \sim (m_2, n_2) \text{ if } m_1 + n_2 = m_2 + n_1$$

with its obvious group structure. If M is finite and integral it is already a group.

2. $A \hookrightarrow A \otimes_{\mathbb{Z}} \mathbb{Q}$ iff A is torsion free.

3. M f.g. $\Rightarrow < M >$ f.g. and A f.g. $\Rightarrow A \otimes_{\mathbb{Z}} \mathbb{Q}$ f.g. but not vice versa.

4. There is a bijection

$$\left\{ \begin{array}{c} \text{f.g. saturated submonoids } M \text{ of } \mathbb{Z}^n \\ (\text{s.t. } < M >= \mathbb{Z}^n) \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{rational polyhedral cones in } \mathbb{R}^n \\ (\text{not containing a line}) \end{array} \right\}$$

$$\begin{array}{rcl} M & \mapsto & M^\vee \\ \sigma^\vee \cap \mathbb{Z}^n & \leftarrow\!\shortmid & \sigma \end{array}$$

5. Products of fine monoids are fine.

6. Equalizers (in particular kernels) of maps from fine monoids to integral ones are fine.

7. Fibre products (in particular intersections) of fine monoids over integral ones are fine.

Proof. 1.–3. are obvious. The only point in 4. is that $\sigma^\vee \cap \mathbb{Z}^n$ is finitely generated. This is called Gordon's Lemma and seen as follows: Let $v_1, \ldots, v_n \in \mathbb{Z}^n$ be generators of $\sigma^\vee$ and let $x \in \sigma^\vee \cap \mathbb{Z}^n$ be given. We may write:

$$x = n_1 v_1 + \cdots + n_n v_n + x_0,$$

with $n_i \in \mathbb{N}$ and where

$$x_0 = q_1 v_1 + \cdots + q_n v_n$$

with $q_i \in \mathbb{Q} \cap [0,1]$. This shows that $\sigma^\vee \cap \mathbb{Z}^n$ is generated by the *finite* set

$$\mathbb{Z}^n \cap \sum_i [0,1] v_i.$$

5. is obvious. 6. The equalizer of two maps $\alpha, \beta \in \mathrm{Hom}(M,N)$ is also the equalizer of the composition $\overline{\alpha}, \overline{\beta} \in \mathrm{Hom}(M, < N >)$ because N is integral, hence it is the kernel of $\overline{\alpha} - \overline{\beta}$. Therefore it suffices to see the finite generation of the kernel of a morphism $\rho : M \to A$ to an Abelian group. Consider a diagram with exact rows:

$$\begin{array}{ccccc} \ker(\rho \circ \mu) & \hookrightarrow & \mathbb{N}^r & \xrightarrow{\rho \circ \mu} & A \\ \downarrow & & \downarrow \mu & & \| \\ \ker(\mu) & \hookrightarrow & M & \xrightarrow{\rho} & A \end{array}$$

A diagram chase shows that the left vertical arrow is surjective. This reduces to show the finite generation of the kernel of a morphism $\gamma : \mathbb{N}^r \to A$. Now look at the following diagram with exact rows:

$$\begin{array}{ccccc} \ker(\gamma) & \hookrightarrow & \mathbb{N}^r & \xrightarrow{\gamma} & A \\ \downarrow & & \downarrow & & \| \\ \ker(\overline{\gamma}) & \hookrightarrow & \mathbb{Z}^n & \xrightarrow{\overline{\gamma}} & A \end{array}$$

A diagram chase shows that

$$\ker(\gamma) = \ker(\overline{\gamma}) \cap \mathbb{N}^r = ((\mathbb{R}_{\geq 0})^r \cap \ker(\overline{\gamma})_{\mathbb{R}}) \cap \ker(\overline{\gamma}),$$

which is finitely generated by 4. (Gordon's Lemma). Finally 7. follows from 5. and 6. $\square$

2 Graded rings and sheaves

Definition 2.1. *Let R be a commutative ring and M an Abelian monoid. A commutative R-algebra B together with a decomposition*

$$B = \bigoplus_{m \in M} B_m$$

*such that $B_m \cdot B_n \subseteq B_{m+n}$ is called M-**graded**. The same definition for a ringed space $(X, \mathcal{O}_X)$ and a sheaf B of $\mathcal{O}_X$-algebras.*
This defines categories, where morphisms are supposed to be homogenous.

2.2. We have the adjunction:

$$[\ \textbf{comm } R\textbf{-alg}\ ] \underset{M \mapsto R[M]}{\overset{\text{mult. monoid}}{\rightleftarrows}} [\ \textbf{Ab mon}\ ]$$

Here $R[M]$ is the ring defined by $\bigoplus_{m \in M} R[m]$ and the multiplication $[m_1][m_2] = [m_1 + m_2]$.

2.3. If M is a *group* and R, S comm. rings, we have

$$\mathrm{Hom}_{\mathrm{spec}(R)}(\mathrm{spec}(S), \mathrm{spec}(R[M])) = \mathrm{Hom}_R(R[M], S) = \mathrm{Hom}_{group}(M, S^*)$$

Since the latter are Abelian groups in a functorial way, $\mathrm{spec}(R[M])$ is a group scheme over $\mathrm{spec}(R)$.

Example 2.4. $A = \mathbb{Z}/n\mathbb{Z}$*:*

$$\mathrm{Hom}_{\mathrm{spec}(R)}(\mathrm{spec}(S), \mathrm{spec}(R[A])) = \{x \in S^* \mid x^n = 1\},$$

i.e. $\mathrm{spec}(R[A]) = \mu_{n,R}$.
$M = \mathbb{Z}$*:*

$$\mathrm{Hom}_{\mathrm{spec}\, R}(\mathrm{spec}(S), \mathrm{spec}(R[A])) = S^*,$$

i.e. $\mathrm{spec}(R[A]) = \mathbb{G}_{m,R}$.

In general, by the structure theorem of Abelian groups, $\mathrm{spec}(\mathbb{R}[A])$ is a product of these group schemes. They are called split **diagonalizable group schemes** or split **quasi-tori**.

2.5. Recall that an action of an R-group scheme G on an R-scheme X is a morphism

$$G \times_{\mathrm{spec}(R)} X \to X$$

over $\mathrm{spec}(R)$ which satisfies the axioms of an action.

Proposition 2.6. *Let A be an Abelian group and R be a commutative ring. There is a 1-1 correspondence*

$$\left\{ A\text{-graded } R\text{-algebras} \right\} \longrightarrow \left\{ \begin{array}{c} \text{affine schemes over } \mathrm{spec}(R) \\ \text{with a } \mathrm{spec}(R[A])\text{-action} \end{array} \right\}$$

Proof. An action as in the RHS is given by an R-algebra-hom

$$B \xrightarrow{\alpha} B \otimes_R R[A]$$

(say given by $b \mapsto \sum_{m \in A} \alpha_m(b)[m]$) satisfying

1. $$B \xrightarrow{\alpha} B \otimes_R R[A] \xrightarrow{\text{counit}} B$$

   is the identity

2.

$$\begin{array}{ccc} B & \xrightarrow{\alpha} & B \otimes_R R[A] \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\ B \otimes_R R[A] & \xrightarrow{\text{comult.}} & B \otimes R[A] \otimes R[A] \end{array}$$

is commutative.

1. boils down to $\sum_m \alpha_m(b) = b$ and 2. to

$$\alpha_n(\alpha_m(b)) = \begin{cases} \alpha_m(b) & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

Since α is a ring-hom, we get a grading

$$B = \bigoplus_m \alpha_m(B).$$

Conversely, given an A-graded ring

$$B = \bigoplus_{m \in A} B_m$$

define

$$\alpha(b) = \sum_m b_m[m]$$

which satisfies 1. and 2. above. □

2.7. Let $\varphi : M \to N$ be a morphism of monoids. There is an adjunction:

$$[\ M\textbf{-graded } R\textbf{-alg}\ ] \underset{\varphi^*}{\overset{\varphi_*}{\rightleftarrows}} [\ N\textbf{-graded } R\textbf{-alg}\ ].$$

In other words, we have

$$\mathrm{Hom}_M(C, \varphi^* B) = \mathrm{Hom}_N(\varphi_* C, B).$$

The functors are defined as follows: If $C = \bigoplus_{m \in M} C_m$ is an M-graded algebra, we define an N grading on $\varphi_* C = C$ by $(\varphi_* C)_n = \bigoplus_{m \in \varphi^{-1}(n)} C_m$.
If $B = \bigoplus_{n \in N} B_n$ is an N-graded algebra, we define $\varphi^* B = \bigoplus_{m \in M} B_{\varphi(m)}$ with the obvious multiplication.

Lemma 2.8.

1. *If φ is injective, we have $\varphi_* \varphi^* B \hookrightarrow B$. The left hand side is called the* **Veronese subring**. *We have furthermore $C \cong \varphi^* \varphi_* C$.*
2. *If φ is surjective, we have $\varphi_* \varphi^* B \twoheadrightarrow B$.*

Example 2.9. *Let $\varphi : M \hookrightarrow N$ be injective and C an M-graded algebra. $\varphi_* C$ is called the* **extension by 0**.

Example 2.10. *Let $\varphi : \mathbb{Z} \to \mathbb{Z}$ be multiplication by n. Consider the ring $B = k[x_1, \ldots, x_n]$ with its natural $\mathbb{Z}$-grading. The Veronese subring $\varphi^* B$ is the subring of B generated by monomials of degree n. Both rings are the graded rings corresponding to a projective space and the associated morphism is the Veronese embedding.*

Example 2.11. *Let $\varphi : M \to 0$ the trivial map and $B = R$ the trivial 0-graded ring. We have $\varphi^* B \cong R[M]$.*

The following is crucial for the proof of finite generation of graded rings:

Proposition 2.12. *Let $\varphi : M \to N$ be a morphism of fine monoids, and B a N-graded R-algebra. We have*

1. *B f.g. $\Rightarrow \varphi^* B$ f.g.*

2. *Assume that R is excellent and B is integral. If for all $n \in N$, $B_n \neq 0$ and there is $d > 0$ s.t. $dn \in \varphi(M)$ then*[1]

   $\varphi^* B$ *f.g.* $\Rightarrow B$ *f.g.*

Proof. 1. B is generated over R by $f_1, \ldots, f_r$, w.l.o.g. homogenous. This induces a homogenous morphism

$$\rho_*(R[\mathbb{N}^r]) \xrightarrow{f} B,$$

where $\rho : \mathbb{N}^r \to N$ is a homomorphism (given by the degrees of the f_i). Consider a Cartesian diagram:

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{N}^r \\ \downarrow & & \downarrow \rho \\ M & \xrightarrow{\varphi} & N \end{array}$$

Proposition 1.3 shows that X is finitely generated. We can define a morphism

$$R[X] \to \varphi^* B$$

by sending $[z, m]$ (with $\rho(z) = \varphi(m)$) to $f(z)$ sitting in $(\varphi^* B)_m$. Since f maps $(\rho_* R[\mathbb{N}^r])_n$ surjectively to B_n, this map is surjective. Therefore $\varphi^* B$ is finitely generated.
2. We may factor $M \twoheadrightarrow \mathrm{im}(\varphi) \hookrightarrow N$. If φ is surjective, we have $\varphi^* B \twoheadrightarrow B$, hence B is finitely generated. This reduces to the case φ injective, hence $\varphi^* B$ can be considered as a (Veronese) subring of B. Choose non-zero homogenous elements $f_1, \ldots, f_n$ such their degrees generate N. For each i, we have $f_i^d \in \varphi^* B$ for some d by assumption. Hence we have a homomorphism of fields

$$\mathrm{Quot}(\varphi^* B)[f_1, \ldots, f_n] \to \mathrm{Quot}(B)$$

where the left hand side is algebraic, hence finite over $\mathrm{Quot}(\varphi^* B)$. Let $x \in B$ be a given homogenous element. We have

$$x \cdot f_1^{k_1} \cdots f_n^{k_n} \in \varphi^* B$$

for some $k_1, \ldots, k_n$ by the choice of the f_i. This shows that the above homomorphism is in fact an isomorphism. Since B is integral over $\varphi^* B$ (because $f_i^d \in \varphi^* B$ for some d) it is a $\varphi^* B$-submodule integral closure of $\varphi^* B$ in the finite field extension $\mathrm{Quot}(B)$, which is a finitely generated $\varphi^* B$-module [EGA IV, 7.8.3 (vi)]. It is therefore, as a submodule, finitely generated itself. Above we used that R is excellent, hence $\varphi^* B$ is excellent because it is finitely generated over R. ☐

Example 2.13. *Let $R = k$ a field. If $\varphi : M \to N$ was actually a morphism of finitely generated Abelian groups, we let $T_1 = \mathrm{spec}(k[M])$ and $T_2 = \mathrm{spec}(k[N])$ be the associated quasi-tori. In this case:*

$$\mathrm{spec}(\varphi^* B) = (\mathrm{spec}(B) \times_k T_1)/T_2,$$

where T_2 acts on both factors. In the RHS we understand the categorical quotient. And

$$\mathrm{spec}(\varphi_* C)$$

is $\mathrm{spec}(C)$ *considered as scheme with T_2-action by composition with* $\mathrm{spec}(k[\varphi]) : T_2 \to T_1$.

2.14. Let M and N be finitely generated Abelian groups and let $\varphi : M \twoheadrightarrow N$ be a surjection. We would like to know under which circumstances an M-graded ring C is of the form $\varphi^* B$ and whether the ring B is uniquely determined by this. Here is the criterion:

[1] If φ is surjective, the assertion of 2. is trivial and no assumptions are needed.

Proposition 2.15. *We have $C = \varphi^* B$ for some N-graded R-algebra B, if and only if* $\ker^* C \cong C_0[\ker(\varphi)]$ *(homogenous C_0-isomorphism) such that for all $m \in M$, $k \in \ker(\varphi)$ the multiplication*

$$C_m \otimes_{C_0} C_k \to C_{k+m}$$

is an isomorphism.
The ring B is uniquely determined up to isomorphism if C_0^* is divisible by the exponent of N_{tors}. (otherwise it may depend on the choice of isomorphism above).
The same assertion is true for sheaves on a ringed space $(X, \mathcal{O}_X)$, if in the last condition C_0^* is replaced by $H^0(X, C_0^*)$.

Proof. The only if part is clear from the definition of φ^*. By abuse of notation, denote by $[k]$ for $k \in \ker(\varphi)$ the preimage of $[k]$ under the isomorphism above. We define the ring B by defining B_n as $\bigoplus_{m \in \varphi^{-1}(n)} C_m$ but identifying C_x with C_{x+k} by multiplication with $[k]$ (which is an isomorphism by assumption). This defines obviously a graded ring and we can define an isomorphism

$$C \to \varphi^* B = \bigoplus_{m \in M} B_{\varphi(m)}$$

by mapping a homogenous element x of degree m to the projection onto B but cosidering it in the m'th summand of the sum.
Two such constructions B and B_χ differ by a homogenous C_0-automorphism of $C_0[\ker(\varphi)]$ which is obviously given by a character $\chi : \ker(\varphi) \to C_0^*$. If C_0 is divisible by the exponent of N_{tors}, we may lift the character to a character $\chi' : M \to C_0^*$ and define an graded C_0-automorphism

$$C \to C$$

on homogenous elements by $c_m \mapsto \chi'(m) c_m$. This induces an isomorphism between B and B_χ. □

3 Divisorial rings and sheaves

Definition 3.1. *Let X be an integral variety over $k = \overline{k}$ and M a f.g. submonoid of* $\mathrm{Div}_{\mathbb{Q}}(X)$. *The sheaf of $\mathcal{O}_X$-algebras*

$$\mathcal{R}(X; M) = \bigoplus_{D \in M} \mathcal{O}(\lfloor D \rfloor)$$

where

$$\mathcal{O}(D)(U) = \{ f \in K(X) \mid |\mathrm{div}(f) + D|_U \geq 0 \}$$

w.r.t. the multiplication inherited from $K(X)$ is called the **divisorial sheaf** *associated with M. Similarly the ring of its global sections*

$$R(X; M) = \bigoplus_{D \in M} H^0(X, \mathcal{O}(\lfloor D \rfloor))$$

is called the **divisorial ring** *associated with M.*

Definition 3.2. *A special role is played by those divisorial rings in which M is generated by divisors D_i which are rationally equivalent to a rational positive multiple of $K_X + \Delta_i$, where Δ_i is effective. They are called* **adjoint** *rings.*

3.3. The main question is whether the divisorial rings are finitely generated. In contrast the divisorial sheaf is of finite type under pretty general conditions, for example if X is locally $\mathbb{Q}$-factorial. We will later in the seminar see the proof of the following:

Theorem 3.4. *Let X be a smooth projective variety and let Δ be a $\mathbb{Q}$-divisor with simple normal crossings such that $\lfloor D \rfloor = 0$. Then the* **log canonical ring**

$$R(X; K_X + \Delta)$$

is finitely generated.

Its proof requires the consideration of divisorial rings associated with monoids M other than $\mathbb{N}$. A typical intermediate step is a theorem of the form:

Theorem 3.5. *Let X be a smooth projective variety of dimension n. Let $B_1, \dots, B_k$ be $\mathbb{Q}$-divisors on X such that $\lfloor B_i \rfloor = 0$ for all i, and such that the support of $\sum_{k=1}^{k} B_i$ has simple normal crossings. Let A be an ample $\mathbb{Q}$-divisor on X, and denote $D_i = K_X + A + B_i$ for every i. Then the adjoint ring*

$$R(X; D_1, \dots, D_k)$$

is finitely generated.

4 Cox rings and sheaves

Let again X be an integral variety over $k = \overline{k}$ and M a f.g. submonoid of $\mathrm{Div}(X)$. From the definition of the divisorial sheaves and rings it is to be expected that the divisorial sheaf/ring only depends on the image of M in the class group $\mathrm{Cl}(X)$.

Proposition 4.1. *Let $\varphi : M \twoheadrightarrow \mathrm{im}(M) \subseteq \mathrm{Cl}(X)$ the projection. Then there is an $\mathrm{im}(M)$-graded sheaf $\widetilde{\mathcal{R}}$ such that*

$$\mathcal{R}(X; M) \cong \varphi^* \widetilde{\mathcal{R}}.$$

(similarly for $R(X; M)$).
If either $H^0(X, \mathcal{O}_X^*) = k^*$ or $\mathrm{Cl}(X)$ is torsion-free, $\widetilde{\mathcal{R}}$ is uniquely determined up to isomorphism.

Definition 4.2. *If $\varphi(M) \to \mathrm{Cl}(X)$ is surjective (in particular $\mathrm{Cl}(X)$ is f.g.) and the assumptions above are satisfied, the sheaf $\widetilde{\mathcal{R}}(X) = \widetilde{\mathcal{R}}$ is called the* **Cox sheaf** *of X. By the uniqueness it does not depend on M either (up to isomorphism)*[2]*. The ring of its global section $\widetilde{R}(X)$ is called the* **Cox ring** *of X.*

4.3. The Cox sheaf and Cox ring are $\mathrm{Cl}(X)$-graded by construction or equivalently $\mathbf{spec}_X(\widetilde{\mathcal{R}}(X))$ is equipped with an action of $\mathrm{spec}(k[\mathrm{Cl}(X)])$ (the **characteristic quasi-torus**) acting fibrewise. Similarly $\mathrm{spec}(\widetilde{R}(X))$ is equipped with a $\mathrm{spec}(k[\mathrm{Cl}(X)])$ action. In general we have

$$\mathcal{R}(X; M) = \varphi^* \widetilde{\mathcal{R}}(X) \qquad R(X; M) = \varphi^* \widetilde{R}(X)$$

for any divisorial sheaf/ring. In particular, if the Cox ring is finitely generated, *all* divisorial rings are finitely generated. If X is, in addition, normal of affine intersection we have a diagram

$$\begin{array}{ccc} \mathbf{spec}_X(\widetilde{\mathcal{R}}(X)) & \hookrightarrow & \mathrm{spec}(\widetilde{R}(X)) \\ \downarrow & & \\ X & & \end{array}$$

where the horizontal arrow is an equivariant open embedding (with complement of codim ≥ 2) and the vertical arrow can be identified with the categorical quotient of the action of the characteristic quasi-torus. If X is $\mathbb{Q}$-factorial it is a geometric quotient in the sense of [GIT]. If X is locally factorial, the action is free.

[2] For M and M' look at the Cartesian diagram

$$\begin{array}{ccccc} & & M'' & & \\ & \swarrow & & \searrow & \\ M & & & & M' \\ & \searrow & & \swarrow & \\ & & \mathrm{Cl}(X) & & \end{array}$$

Proof of the Proposition. We have to show that the criteria of Proposition 2.15 are satisfied. Let ker : $\ker(\varphi) \hookrightarrow M$ be the inclusion. We have obviosly

$$\ker^* \mathcal{R}(X;M) = \mathcal{R}(X;\ker(\varphi)).$$

Hence we have to see that for a divisorial sheaf associated with a submonoid N of *rational* divisors there is an isomorphism

$$\mathcal{R}(X;N) \cong \mathcal{O}_X[N].$$

This has to be given by a homomorphism $\alpha : N \to K(X)$ such that $\operatorname{div}(\alpha(D)) = D$ which can clearly be chosen. The second assertion is that

$$\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(\operatorname{div}(f)) \to \mathcal{O}_X(D + \operatorname{div}(f))$$

be isomorphisms which is obvious, however. The same argument works for the rings instead of sheaves. □

5 Finite generation of divisorial rings associated with semi-ample bundles

Let X be a projective smooth variety over $k = \overline{k}$ and $\underline{D} = (D_1, \ldots, D_r)$ be a tuple of divisors. If $\underline{n} = (n_1, \ldots, n_r)$ we will write $\underline{nD}$ for the divisor $n_1D_1 + \cdots + n_rD_r$. We write $\underline{n} > d$ if all $n_i > d$.

Proposition 5.1 (Zariski)**.** *If $\underline{D}$ consists of semi-ample divisors then the section ring*

$$R(X; D_1, \ldots, D_r) = \bigoplus_{\underline{n} \in \mathbb{N}^r} H^0(X, \mathcal{O}(\underline{nD}))$$

is finitely generated.

Proof. It is clear that the section ring is integral. Using Proposition 2.12 we may assume that $D_1, \ldots, D_r$ are actually generated by global sections (i.e. base point free). We are furthermore reduced to show that the multiplication

$$H^0(X, \underline{n}_1\underline{D}_1) \times H^0(X, \underline{n}_2\underline{D}_2) \to H^0(X, \underline{n}_1\underline{D}_1 + \underline{n}_2\underline{D}_2)$$

is surjective, provided $\underline{n}_1 > d$, $\underline{n}_2 > d$. Here $\underline{D}_1$ and $\underline{D}_2$ are arbitrary subsets of the original set of divisors. Let $\varphi_{j,i}$ be the morphism

$$\varphi_{j,i} : X \to \mathbb{P}^N$$

determined by the sections in $H^0(X, \mathcal{O}(D_{j,i}))$ and let φ_j $(j = 1, 2)$ the product over all i

$$\varphi_j : X \to (\mathbb{P}^N)^{r_j}$$

(we may assume that always the same N occurs). We have then

$$\mathcal{O}(\underline{n}_j\underline{D}_j) = \varphi_j^* \mathcal{O}(\underline{n})$$

and hence

$$H^0(X, \underline{n}_j\underline{D}_j) = H^0((\mathbb{P}^N)^{r_j}, (\varphi_j)_*(\varphi_j)^*\mathcal{O}(\underline{n}_j)) = H^0((\mathbb{P}^N)^{r_j}, ((\varphi_j)_*\mathcal{O}_X) \otimes \mathcal{O}(\underline{n}_j))$$

(in the second step, we used the projection formula).
Denote by Y_j the image of φ_j. Consider the 'diagonal':

$$\varphi_1 \times \varphi_2 : X \to (\mathbb{P}^N)^{r_1+r_2}$$

and denote Y its image.
According to Lemma 5.3 by replacing $\underline{D}_j$ with some multiple again we may assume that $\varphi_{j*}\mathcal{O}_X = \mathcal{O}_{Y_j}$ and $(\varphi_1 \times \varphi_2)_*\mathcal{O}_X = \mathcal{O}_Y$.

Consider now the exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{Y_1\times Y_2} \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

on $(\mathbb{P}^N)^{r_1+r_2}$. Tensoring it with $\mathcal{O}(\underline{n}_1,\underline{n}_2)$, we get

$$0 \longrightarrow \mathcal{I}\otimes\mathcal{O}(\underline{n}_1,\underline{n}_2) \longrightarrow \mathcal{O}_{Y_1\times Y_2}\otimes\mathcal{O}(\underline{n}_1,\underline{n}_2) \longrightarrow \mathcal{O}_Y\otimes\mathcal{O}(\underline{n}_1,\underline{n}_2) \longrightarrow 0$$

We get the long exact sequence of cohomology

$$H^0((\mathbb{P}^N)^{r_1+r_2},\mathcal{O}_{Y_1\times Y_2}\otimes\mathcal{O}(\underline{n}_1,\underline{n}_2)) \longrightarrow H^0((\mathbb{P}^N)^{r_1+r_2},\mathcal{O}_Y\otimes\mathcal{O}(\underline{n}_1,\underline{n}_2))$$

$$\longrightarrow H^1((\mathbb{P}^N)^{r_1+r_2},\mathcal{I}\otimes\mathcal{O}(\underline{n}_1,\underline{n}_2))$$

By Lemma 5.4 we get the vanishing of the H^1 for $\underline{n}_1 > d$, $\underline{n}_2 > d$. But

$$H^0((\mathbb{P}^N)^{r_1+r_2},\mathcal{O}_{Y_1\times Y_2}\otimes\mathcal{O}(\underline{n}_1,\underline{n}_2)) = H^0(X,\mathcal{O}_X(\underline{n}_1\underline{D}_1))\otimes H^0(X,\mathcal{O}_X(\underline{n}_2\underline{D}_2))$$

and

$$H^0((\mathbb{P}^N)^{r_1+r_2},\mathcal{O}_Y\otimes\mathcal{O}(\underline{n}_1,\underline{n}_2)) = H^0(X,\mathcal{O}_X(\underline{n}_1\underline{D}_1+\underline{n}_2\underline{D}_2))$$

and the map is multiplication. The statement follows. □

Corollary 5.2. *If $D_1,\dots,D_r$ are as in the Proposition and $M\subset \mathrm{Div}(X)$ is the submonoid generated by them, then the divisioral ring*

$$R(X;M)$$

associated with M is finitely generated.

Proof. Apply Proposition 2.12 to the morphism $\mathbb{N}^r \twoheadrightarrow M$. □

Lemma 5.3. *Let D be a semi-ample divisor on X. For some integer n the morphism*

$$\varphi_{nD}: X\to\mathbb{P}^N$$

determined by $\mathcal{O}(nD)$ satisfies $(\varphi_{nD})_*\mathcal{O}_X = \mathcal{O}_Y$, where Y is the image.

Proof. W.l.o.g. we assume that D is generated by global sections. We have the Stein factorization of φ_D [Hartshorne, III, Corollary 11.7]:

$$X \xrightarrow{\ \varphi\ } X' \xrightarrow{\ p\ } \mathbb{P}^N$$

and

$$H^0(X,\mathcal{O}(nD)) = H^0(X,\varphi_D^*\mathcal{O}(n)) \cong H^0(X',(\varphi_*\mathcal{O}_X)\otimes p^*\mathcal{O}(n)) = H^0(X',p^*\mathcal{O}(n)).$$

Here we used $\varphi_*\mathcal{O}_X = \mathcal{O}_{X'}$. This means that φ_{nD} factors through φ. Now $p^*\mathcal{O}(n)$ is very ample for some n [Hartshorne, III, Exercise 5.7 (d)]. Hence it induces an embedding of X' into some $\mathbb{P}^{N'}$. □

Lemma 5.4. *Let $X = (\mathbb{P}^N)^r$ and $\mathcal{L}$ a coherent sheaf on X. There is an integer d such that*

$$H^i(X,\mathcal{L}\otimes\mathcal{O}(\underline{n})) = 0$$

for all $i>0$, $\underline{n}>d$.

Proof. This is a refinement of Serre's vanishing Theorem [Hartshorne, III, Theorem 5.2] and proven the same way. □

5.5. Finally a counterexample (taken from [CaL2]): Let E be an elliptic curve and let D be a divisor of degree 0 which it not torsion. Let $B_2 \in \mathrm{Div}_{\mathbb{Q}}$ be a divisor of non-zero degree with $\lfloor B_2 \rfloor = 0$. Then

$$R(E; K_E + D, K_E + D + B_2) = \bigoplus_{\underline{n} \in \mathbb{N}^2} H^0(E, \mathcal{O}(\lfloor (n_1 + n_2)D + n_2 B_2 \rfloor))$$

is *not* finitely generated. To show this (Prop. 2.12), it suffices to assume that D and $D + B_2$ are integral. If R would be finitely generated,

$$M = \{\underline{n} \in \mathbb{N}^2 \mid H^0(E, \mathcal{O}((n_1 + n_2)D + n_2 B_2)) \neq 0\}$$

would be a finitely generated submonoid. By Riemann-Roch we have

$$H^0(E, \mathcal{O}(n_1 D)) = 0$$

for $n_1 > 0$ but

$$H^0(E, \mathcal{O}((n_1 + n_2)D + n_2 B_2) \neq 0$$

for $n_2 > 0$ because $(n_1 + n_2)D + n_2 B_2$ has positive degree. Hence M is not finitely generated.

6 Fixed parts

Let X be a smooth projective variety.
For a divisor $D \in \mathrm{Div}(X)$ we denote

$$\mathrm{Fix}(D) = \min_{E \in |D|} E$$

(where is minimum is taken component-wise) and for $D \in \mathrm{Div}_{\mathbb{Q}}(X)$:

$$\mathbf{Fix}(D) = \liminf_{m > 0} \frac{1}{m} \mathrm{Fix}(mD)$$

for $m \in \mathbb{N}$ sufficiently divisible.

Proposition 6.1. *Let $D_1, \ldots, D_n \in \mathrm{Div}_{\mathbb{Q}}(X)$ be given and assume that*

$$R(X; D_1, \ldots, D_n)$$

is finitely generated. Let M be the submonoid of $\mathrm{Div}_{\mathbb{Q}}(X)$ generated by $D_1, \ldots, D_n$.

1. *The function* **Fix** *extends to a piecewise linear function on $\mathbb{R}_{\geq 0} M \subseteq \mathrm{Div}_{\mathbb{R}}(X)$.*
2. *There is an integer k such that for all $D \in kM$ we have $\mathbf{Fix}(D) = \mathrm{Fix}(D)$*

Proof. 1. We have a (w.l.o.g. homogenous) morphism

$$\rho : \varphi_*(k[\mathbb{N}^r]) \to R(X; M)$$

where $\varphi : \mathbb{N}^r \to M$ is a homomophism. Let $\widetilde{\varphi} : \mathbb{R}^r_{\geq 0} \to \mathbb{R}_{\geq 0} M \subset < M >_{\mathbb{R}}$ be the extension.
For each $D \in M$ we have

$$\mathrm{Fix}(D) = \min_{n \in \varphi^{-1}(D)} \mathrm{div}(\rho([n]))$$

and

$$\mathbf{Fix}(D) \quad = \quad \liminf_{m} \frac{1}{m} \min_{n \in \varphi^{-1}(mD)} \mathrm{div}(\rho([n])).$$

Now $\operatorname{div} \circ \rho$ is a linear function $l : \mathbb{N}^r \to \operatorname{Div}(X)^+$ provided the argument is sufficiently divisible. We may write:

$$\mathbf{Fix}(D) = \liminf_m \min_{n \in \widetilde{\varphi}^{-1}(D) \cap \frac{1}{m}\mathbb{N}^r} l(n).$$

Therefore

$$\mathbf{Fix}(D) = \min_{n \in \widetilde{\varphi}^{-1}(D)} l(n).$$

This makes already sense for $D \in \mathbb{R}_{\geq 0} M$. To show that it defines a piecewise linear function on $\mathbb{R}_{\geq 0} M$, it suffices to show that finitely many functions $\operatorname{mult}_G \circ l$ for prime divisors G are piecewise linear. But in this case the statement follows from Lemma 6.2.

2. It suffices to show this on one of the sub-cones $\mathbb{R}_{\geq 0} M'$ where **Fix** is linear. Here M' is any f.g. submonoid of $\operatorname{Div}(X)$ generating this cone. If k is sufficiently large, $D \in kM$ will ensure $k \in M'$ and

$$\mathbf{Fix}(D) = \mathbf{Fix}(\sum \alpha_i m_i) = \sum \alpha_i \mathbf{Fix}(m_i)$$

Now the minimum in $\mathbf{Fix}(m_i)$ is actually attained on the intersection of $\widetilde{\varphi}^{-1}(m_i)$ with a face of $\mathbb{R}^r_{\geq 0}$. This intersection contains a rational point. This means that $\mathbf{Fix}(dm_i) = \operatorname{Fix}(dm_i)$ for some d. Taking k to be the l.c.m. of these d, we have

$$\mathbf{Fix}(D) = \operatorname{Fix}(D)$$

provided $D \in kM'$. □

Lemma 6.2. *Let $\sigma \subset \mathbb{R}^r$ be a rational polyhedral cone not containing a line, $\rho : \mathbb{R}^r \twoheadrightarrow \mathbb{R}^m$ be a projection and $\alpha : \mathbb{R}^r \to \mathbb{R}$ be a linear form which is non-negative on σ. Then the function*

$$\begin{aligned} \rho(\sigma) &\to \mathbb{R}_{\geq 0} \\ m &\mapsto \min_{x \in \rho^{-1}(m) \cap \sigma} \alpha(x) \end{aligned}$$

is piecewise linear.

Proof. Sketch: $\rho(\sigma)$ is covered by the isomorphic images of faces of σ of appropriate dimensions. The min is always attained on one of these faces. □