Compactifications of moduli spaces and mixed Hodge structures - with applications to automorphic forms

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14. Juli 2008

1 Introduction

Main references for this talk are:

[Milne] J. S. Milne, *Canonical Models of (Mixed) Shimura Varieties and Automorphic Vector Bundles*, can be obtained from http://www.jmilne.org.

[Pink] R. Pink, *Arithmetical compactification of mixed Shimura varieties*, Dissertation (1989), Bonner Mathematische Schriften 209

See also:

[AMRT] A. Ash, D. Mumford, M. Rapoport, Y. S. Tai, *Smooth compactification of locally symmetric varieties*, Math. Sci. Press, Brookline MA, 1975.

[Brylinski] J.-L. Brylinski, *1-motifs et formes automorphes*, Journées automorphes, 43-106, Publ. math. Univ. Paris VII (1983)

[Deligne, HodgeII] P. Deligne, *Théorie de Hodge, II*, Publications Mathématiques de l'IHÉS, 40 (1971), p. 5-57

[Deligne, HodgeIII] P. Deligne, *Théorie de Hodge, III*, Publications Mathématiques de l'IHÉS, 44 (1974), p. 5-77

[FC] G. Faltings, C.-L. Chai *Degeneration of Abelian Varities*, Springer (1990)

(1.0.1) We have seen that abelian varieties over $\mathbb{C}$ can be parameterized via their 'periods'. Recall, if (A, Ψ) is a polarized complex abelian variety of dimension g, they are given by the following construction: Choose a symplectic[1] basis $v_1, \dots, v_{2g}$ of $\pi_1(A, 0) = H_1(A, \mathbb{Z})$, with $\langle v_i, v_{j+g} \rangle = -\langle v_{j+g}, v_i \rangle = \delta_{ij}$.
One can choose a unique basis $\omega_1, \dots, \omega_g$ of $H^0(A, \Omega^1)$, the space of global holomorphic 1-forms (these are necessarily closed and invariant), such that $\int_{v_i} \omega_j = \delta_{ij}$.
The *matrix of periods* $\tau := \{\int_{v_{i+g}} \omega_j\}_{ij}$ is then symmetric, has positive definite imaginary part and determines the abelian variety uniquely, since the following map is an isomorphism:

$$\begin{array}{ccc} A & \rightarrow & \mathbb{C}^g/\Lambda \\ a & \mapsto & \left\{ \int_0^a \omega_i \right\}_i \end{array}$$

where Λ is the *lattice of periods*, i.e.

$$\left\{ \{ \int_v \omega_i \}_i, v \in H_1(A, \mathbb{Z}) \right\} = \mathbb{Z}^g + \tau \mathbb{Z}^g.$$

To have an invariant description (independent of choices) and to generalize to higher 'periods', it is convenient to translate this into terms of Hodge structures: Integration, as above, defines a map

$$H^0(A, \Omega^1) \rightarrow H^1(A, \mathbb{C}) = \mathrm{Hom}(H_1(A, \mathbb{Z}), \mathbb{C})$$

and similarly for antiholomorphic forms

$$H^0(A, \overline{\Omega}^1) \rightarrow H^1(A, \mathbb{C}),$$

[1] symplectic structure determined by Ψ

which induces a decomposition

$$H^1(A,\mathbb{C}) = H^0(A,\Omega^1) \oplus H^0(A,\overline{\Omega}^1)$$

into maximal isotropic subspaces. (proof: $H^1(A,\mathbb{C})$ can be interpreted as the space of invariant differential forms, i.e. as vectors in the dual of the tangent space, say at 0. The tangent space decomposes into holomophic and antiholomophic tangent space. The above is just the dual decomposition).
Then A can be obtained as the quotient

$$H_1(A,\mathbb{Z})\backslash H_1(A,\mathbb{C})/H^0(A,\Omega^1)^\perp.$$

Conversely, given any symplectic lattice $L \cong \mathbb{Z}^{2g}$ and a decomposition

$$H_\mathbb{C} = H^{1,0} \oplus H^{0,1},$$

with $\overline{H^{0,1}} = H^{1,0}$ (a Hodge structure of weight 1), into maximal isotropic subspaces (with a positivity condition) one obtains a polarized abelian variety

$$H_\mathbb{Z}^*\backslash H_\mathbb{C}^*/(H^{1,0})^\perp.$$

(1.0.2) Remark. The space $H_\mathbb{C}^*/(H^{1,0})^\perp$ is (via projection) isomophic to $H_\mathbb{R}$, and so a Hodge structure as above is equivalent to give a complex structure on $H_\mathbb{R}$. Conversely one gets $H^{l,m}$ as the subspace of $H_\mathbb{R} \otimes \mathbb{C}$, where $z \in \mathbb{C}$ 'operates' as $z^l\overline{z}^m$.

(1.0.3) The group $\mathrm{Sp}(2g,\mathbb{R})$ operates on H and hence on the set of Hodge structures on H with stabilizer K a unitary group. The operation translates into the usual operation on $\mathbb{H}^g$. The identification $\mathbb{H}^g \cong \mathrm{Sp}(2g,\mathbb{R})/K$ defines a complex structure on $\mathrm{Sp}(2g,\mathbb{R})/K$ (indeed even the structure of Hermitian symmetric domain) which is the same which is induced by considering the $L^{1,0}$ (Hodge filtration) as element of the Grassmannian of *all* maximal isotropic subspaces of $L_\mathbb{C}$, which is naturally a complex analytic variety.
There are 2 kinds of important generalizations of this:

i. Parameterizing *arbitrary* Hodge structures, possibly *with special structures*. This corresponds to homogenous spaces under reductive groups. This way, every Hermitian symmetric domain occurs[2].

ii. Associating a kind of Hodge structure (it will be called **mixed**) also to nonproper and possibly singular varieties. Their parameter spaces are homogenous under - no longer reductive - but more general groups.

We will illustrate in this talk, ...

i. ... how these generalizations look like,

ii. ... how their interplay can explain the construction of compactifications,

iii. ... how **canonical models** (over number fields) of the parameter spaces can be *characterized* purely in group theoretical terms.

To illustrate the last point, consider the j-function.
Its Fourier- or q-expansion

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \cdots \qquad q = e^{2\pi i\tau}$$

has rational (even integer) coefficients. It is related to a parameter space of (very simple) *mixed* Hodge structures.
Its values at $\tau = i, \rho, \frac{1+\sqrt{-7}}{2}, \frac{1+\sqrt{-11}}{2}, \cdots$ [3] are rational (even integers). They are related to parameter spaces of pure Hodge structures but with *additional structure*.

[2] but not quite every reductive group
[3] corresponding to lattices of elliptic curves with complex multiplication by rings of integers of class number 1

We will see how the *characterization* of models of both - parameter spaces of pure and of mixed Hodge structures - explain that there is a strong relation, i.e. *every* function having a rational q-expansion has rational values at these numbers[4]. Conversely a function which takes such values at (sufficiently many) imaginary quadratic τ has a rational q-expansion. Analogously this holds true in a modified form for automorphic forms, as we will (maybe) see in the last section.
The q-expansion part of this is also called the **q-expansion principle**.

2 Mixed Hodge structures and their parameter spaces

2.1 Mixed Hodge structures in nature

(2.1.1) For an arbitrary smooth compact complex variety X of dimension n, every cohomology group $H = H^i(X, \mathbb{Z})$ carries a Hodge structure of weight i, i.e. a decomposition

$$H_{\mathbb{C}} = \bigoplus_{p+q=i} H^{p,q}, \tag{1}$$

with $\overline{H^{p,q}} = H^{q,p}$. This can be obtained as follows: The complex

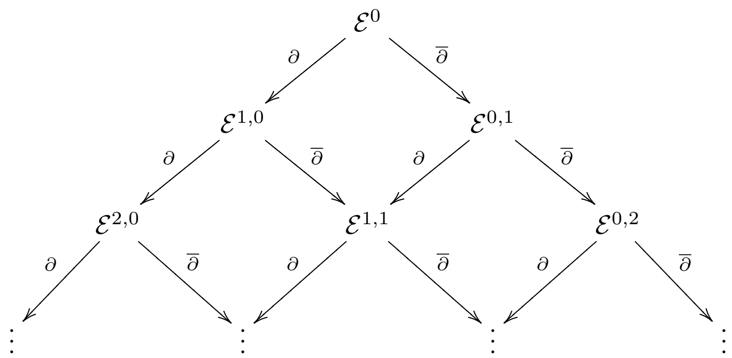
$$0 \longrightarrow \mathcal{O}_X \longrightarrow \Omega^1_X \longrightarrow \cdots \longrightarrow \Omega^n_X \longrightarrow 0 \tag{2}$$

is a resolution of the constant sheaf $\mathbb{C}_X$. The hypercohomology spectral sequence for this resolution defines a filtration $\{F^j(H^i(X, \mathbb{Z}))\}_j$ and one obtains the Hodge decomposition via $H^{l,m} = F^l \cap \overline{F}^m$ ($\{F^j\}_j$ and H can even be calculated using the Zarisky topology!). In more down to earth terms, like in the introduction, this is related to the decomposition of forms into holomorphic and antiholomorphic parts as follows: The sheaf of $\mathcal{C}^\infty$-n-forms $\mathcal{E}^n$ has a decomposition

$$\mathcal{E}^n = \bigoplus_{l+m=n} \mathcal{E}^{l,m}$$

where $\mathcal{E}^{l,m}$ is, locally in a chart with coordinates $z_1, \ldots, z_n$, generated by $\underbrace{dz_? \wedge \cdots \wedge dz_?}_{l \text{ factors}} \wedge \underbrace{d\overline{z}_? \wedge \cdots \wedge d\overline{z}_?}_{m \text{ factors}}$.

There is a double complex



whose total complex

$$0 \longrightarrow \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n \longrightarrow 0$$

is an acyclic resolution of $\mathbb{C}_X$ or the equivalently the complex (2).
Furthermore the complexes

$$0 \longrightarrow \mathcal{E}^{l,0} \xrightarrow{\overline{\partial}} \mathcal{E}^{l,1} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{E}^{l,n-l} \longrightarrow 0$$

[4]possibly in $\mathbb{Q}(\tau)$ - and in the Hilbert-class field if the class number is not 1

are acyclic resolutions of Ω^l. Hence the filtration defined by the interpretation as hypercohomology of (2) is the same defined by the stupid filtration of the total complex, which means just filtering $\mathcal{E}^n$ according to the number of occurring dz's. The Hodge theorem says, that the corresponding spectral sequence degenerates at ${}^2E$, hence the decomposition (1), which the additional information $H^{l,m} \cong H^l(X, \Omega^m)$.

(2.1.2) Now suppose that $X \hookrightarrow \overline{X}$ is a smooth compactification, such that $Y = \overline{X} - X$ is a divisor with normal crossings, which we suppose (for simplicity) to consist of distinct smooth divisors. Denote by $\Omega^p_{\overline{X}} < Y >$ the subsheaf of $j_*\Omega^p_X$ generated locally by $\Omega^1_{\overline{X}}$ and $\frac{dz_i}{z_i}$, if z_i is the local equation of a divisor, component of Y. It has an increasing filtration

$$W'_n(\Omega^p_{\overline{X}} < Y >)$$

according to how many $\frac{dz_i}{z_i}$'s occur at most. Obviously $W'_0(\Omega^p_{\overline{X}} < Y >) = \Omega^p_{\overline{X}}$ and $W'_n(\Omega^p_{\overline{X}} < Y >) = \Omega^p_{\overline{X}} < Y >$ for $n \geq p$. There is a Poincaré residue isomorphism

$$\mathrm{res} : \mathrm{gr}^{W'}_n(\Omega^p_{\overline{X}} < Y >) \cong i_*\Omega^{p-n}_{\widetilde{Y}^n}$$

where $\widetilde{Y}^n$ is the disjoint union of all possible intersections of n distinct divisors. The map associates locally around an intersection like this:

$$\alpha \wedge \frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_n}}{z_{i_n}} \mapsto \pm\alpha|_{\{z_{i_1} = \cdots = z_{i_n} = 0\}},$$

where the sign is defined by an ordering of the divisors. It follows that

$$H^i(\mathrm{gr}^{W'}_n(\Omega^*_{\overline{X}} < Y >)) \cong \begin{cases} i_*\mathbb{C}_{\widetilde{Y}^n} & i = n \\ 0 & i \neq n \end{cases}$$

One can show, that

i. The inclusion $\Omega^*_{\overline{X}} < Y > \hookrightarrow j_*\Omega^*_X$ is a filtered quasiisomorphism for the W'-filtration on $\Omega^*_{\overline{X}} < Y >$ and the canonical filtration on $j_*\Omega^*_X$ respectively.

ii. The complex Ω^*_X is an *acyclic* resulution for $\mathbb{C}_X$ for the functor j_*.

From this follows:

$$R^ij_*\mathbb{C}_X \cong H^i(j_*\Omega^*_X) \cong H^i(\Omega^*_{\overline{X}} < Y >) \cong i_*\mathbb{C}_{\widetilde{Y}^i}$$

and that the induced filtration W on $H^i(X, \mathbb{C})$ is the same induced by the Leray spectral sequence for j_*. It hence converges from the page ${}^2E^{i,j} = H^i(i_*\mathbb{C}_{\widetilde{Y}^j})$ to ${}^\infty E^{i,j} = \mathrm{gr}^{W'}_i(H^{i+j}(X, \mathbb{C}))$.
We have now 2 filtrations on $H^i(X, \mathbb{C})$, the Hodge filtration $\{F^i\}_i$ coming from the (stupid) filtration (according to the number of occurring holomorphic dz_i's) of $\mathcal{E}^*$, and $W_n := W'_{n-i}$ coming from the Leray spectral sequence - or equivalently - from the number of occurring $\frac{dz_i}{z_i}$'s in $\Omega^*_{\overline{X}} < Y >$. The main theorem of [Deligne, HodgeII] is that these filtrations together define a **mixed Hodge structre** on $H = H^i(X, \mathbb{Z})$, i.e.

- a decreasing filtration $\{F^i(H_\mathbb{C})\}_i$ of $H_\mathbb{C}$,
- an increasing filtration $\{W_n(H_\mathbb{Q})\}_n$ of $H_\mathbb{Q}$[5].

such that for each n, $\{F^i\}_i$ induces a pure Hodge structure of weight n on $\mathrm{gr}^W_n(H_\mathbb{Q})$.

(2.1.3) Remark. Everything said so far can be done in a relative setting $f : X \to S$. The result is: The W-filtration is locally constant and the F-filtration varies holomorphically on S. Furthermore there is a notion of Griffiths transversality w. r. t. the Gauss-Manin connection on $R^if_*(\mathbb{C}_X)$ as well.
Furthermore in [Deligne, HodgeIII], Deligne extends the construction to singular varieties.

[5] above we have described only the complexification of this

2.2 Weight ≤ 2 - semiabelian varieties - 1-motives

We have seen, that (polarized) pure Hodge structures of weight 1 are particularly easy and correspond to abelian varieties. Next, we will investigate the mixed Hodgestructure on $H^1(G,\mathbb{Z})$ on a semiabelian variety[6]:

$$0 \longrightarrow \mathbb{G}_m^n \longrightarrow G \longrightarrow A \longrightarrow 0$$

We may compactify each $\mathbb{G}_m$ by adding points 0 and ∞, obtaining a compactification $G \hookrightarrow \overline{G}$. For $i = 1$ the cohomology may be simply calculated as:

$$H^1(G,\mathbb{C}) = \frac{\{\alpha \in (\Omega^1_{\overline{G}} < Y > \otimes_{\mathcal{O}_{\overline{G}}} \mathcal{E}^1_{\overline{G}})(\overline{G}) \mid d\alpha = 0\}}{d\mathcal{E}^0(\overline{G})}$$

and

$$W_1(H^1(G,\mathbb{C})) = \frac{\{\alpha \in \mathcal{E}^1_{\overline{G}}(\overline{G}) \mid d\alpha = 0\}}{d\mathcal{E}^0(\overline{G})}$$

since no logarithmic poles are allowed. (There is only this non-trivial filtration step).
Since $\overline{G}$ is a fibration over A with fibre isomorphic to $\mathbb{P}^1(\mathbb{C})^n$ one sees that the filtration is even defined over $\mathbb{Z}$ and we have:

$$W_1(H^1(G,\mathbb{Z})) \cong H^1(A,\mathbb{Z}) \cong \mathbb{Z}^{2g}$$

and

$$\mathrm{gr}_2^W(H^1(G,\mathbb{Z})) \cong H^1(\mathbb{G}_m^n,\mathbb{Z}) \cong \mathbb{Z}^n.$$

In the special case $G = \mathbb{G}_m$ via the last map the dual of a nontrivial cycle on $\mathbb{G}_m = \mathbb{C}^*$ corresponds to the class of the differential form $\frac{1}{2\pi i}dz/z$ (which of course is not in W_1).
Furthermore the only nontrivial step of the Hodge filtration is given by

$$F^1(H^1(G,\mathbb{C})) = H^0(G,\Omega^1_{\overline{G}} < Y >)$$

of dimension $2g+n$ (and $(W_1)_{\mathbb{C}} \cap F^1$ gives the usual pure Hodge structure on $H^1(A,\mathbb{C})$).
There is again an isomorphism:

$$\begin{array}{rcl} G & \to & H^1(G,\mathbb{Z})^* \backslash H^1(G,\mathbb{C})^*/(F^1)^\perp \\ g & \mapsto & \{\omega \mapsto \int_0^g \omega\}, \end{array}$$

hence to every mixed Hodge structure of type $(1,0),(0,1),(1,1)$ (polarized on the W^1-part) we may associate a semiabelian variety. We have already seen in a previous talk that semiabelian varieties play an important role for the boundary of compactifications of the moduli space of abelian varieties. This has a nice explanation in terms of mixed Hodge structures, as we will see in the next section. However also structures of type $(0,0),(1,0),(0,1),(1,1)$ will play a role. These have a quotient $H/W_0(H)$ which is of the type already described, hence correspond to a semiabelian variety.
There is an exact sequence of mixed Hodge structures

$$0 \longrightarrow W_0(H) \longrightarrow H \longrightarrow H/W_0(H) \longrightarrow 0$$

and there is still one nontrivial filtration step $F^1(H)$ such that the quotient gives back the F^1 of $H/W_2(H)$ described earlier; its intersection with $W_0(H)$ is zero.
Dually we have an exact sequence of Hodge structures

$$0 \longrightarrow (H/W_0)^* \longrightarrow H^* \longrightarrow W_0^* \longrightarrow 0$$

This extension is (per definition) an element of $\mathrm{Ext}^1_{MHS}(W_0^*,(H/W_0)^*)$, where W_0 is the trivial pure Hodge structure of weight 0 and some dimension m.

[6]over $\mathbb{C}$ we may assume w.l.o.g., that the torus part is split

We claim that $\mathrm{Ext}^1_{MHS}(W_0^*, (H/W_0)^*) \cong \mathrm{Hom}(W_0(H), G) \cong G^m$: For this consider the diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & (W_0)^*_{\mathbb{Z}} & & \\
 & & & & & & \| & & \\
0 & \longrightarrow & (H/W_0)^*_{\mathbb{Z}} & \longrightarrow & H^*_{\mathbb{Z}} & \longrightarrow & (W_0)^*_{\mathbb{Z}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{(H/W_0)^*_{\mathbb{C}}}{(F^1)^{\perp}} & \longrightarrow & \frac{H^*_{\mathbb{C}}}{(F^1)^{\perp}} & \longrightarrow & \frac{(W_0)^*_{\mathbb{C}}}{(F^1)^{\perp} \cap (W_0)^*_{\mathbb{C}}} = 0 & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & G & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

from which we get a connecting homomorphism $\delta : W_0(H_{\mathbb{Z}})^* \to G$.
Conversely, if we have some map $\alpha : (W_0)^*_{\mathbb{Z}} \to G$, we may consider the map

$$\begin{array}{c}
(W_0)^*_{\mathbb{Z}} \oplus H_1(G, \mathbb{Z}) \\
\Big\downarrow {\scriptstyle (\text{incl.}, \widetilde{\alpha} + \text{incl.})} \\
(W_0)^*_{\mathbb{C}} \oplus H_1(G, \mathbb{C})
\end{array}$$

where $\widetilde{\alpha}$ is some lift of α, and take the pullback F^1 of the Hodge filtration $F^1(H_1(G, \mathbb{C})) + (W_0)^*_{\mathbb{C}}$ by this map. We get a nontirivial extension of the trivial Hogde structure on $(W_0)^*_{\mathbb{Z}}$ by the mixed Hodge structure of G. An *important* corollary of this construction is, that if $G/\alpha((W_0)^*_{\mathbb{Z}})$ happens to be an abelian variety, then the *pure* Hodge structure on $H_1(G/\alpha((W_0)^*_{\mathbb{Z}})) \cong (W_0)^*_{\mathbb{Z}} \oplus H_1(G, \mathbb{Z})$ is (up to isomorphism) given by the filtration F^1 constructed above, *forgetting the weight filtration.*

(2.2.1) Remark. For example it is true, that similarly $\mathrm{Ext}^1_{MHS}(H/W_1, W_1) = \mathrm{Hom}((H/W_1)_{\mathbb{Z}}, A^{\vee})$, if A is the abelian variety corresponding to the pure Hodge structure on W_1. This corresponds to the fact, that a semiabelian variety of torus rank m can be constructed by m points in $Pic^0(A) = A^{\vee}$ by deleting the 0-sections from the line-bundles and taking the direct sum. (Being in Pic^0 ensures homogeneity and hence one may define a structure of group variety on the complement of the 0-section.)

(2.2.2) Summary. We have seen that the category of Hodge structures of type $(0,0), (1,0), (0,1), (1,1)$ (partly polarized) which fixed dimensions is equivalent to the category of pairs (G, α), where G is a semiabelian variety of fixed dimension and torus-rank and $\alpha : \mathbb{Z}^m \to G$ is a homomorphism. Such a pair is called a **1-motive** [Deligne, HodgeIII]. These structures are more symmetric than those of type $(1,0), (0,1), (1,1)$ and there is a notion of polarization for them as well, which for the mixed Hodge structure means analogously, that $H_{\mathbb{Z}}$ is symplectic, W_1 is isotropic, $W_2 = W_1^{\perp}$ and F^1 is maximal isotropic (and such that a positivity condition on W_2/W_1 holds true). For the corresponding 1-motives the notion of polarization is more involved but there is a beautiful symmetric description in [Deligne, HodgeIII].

2.3 Parameter spaces

(2.3.1) Let G be a reductive $\mathbb{Q}$-group acting faithfully on a vector space $H_{\mathbb{Q}}$ and choose a lattice $H_{\mathbb{Z}} \subset H_{\mathbb{Q}}$.

If we fix a pure Hodge structure (given by a filtration $\{F^i(H_{\mathbb{C}})\}_i$) on H of some weight, then $G(\mathbb{R})$ moves the Hodge structure, i.e. the operation respects the Hodge condition. If we ensure, that there is a bilinear form Ψ on H, such that the given Hodge structure is polarized via Ψ, and if G fixes the bilinear form (up to scalars), then the stabilizer group of $\{F^i\}_i$ is *compact* (modulo the center of) $G(\mathbb{R})$ and the orbit $\mathbb{X}$ of Hodge structures is[7] a Hermitian symmetric domain (the complex structure comes from the Borel embedding, see (3.1)).
If G is not the full automorphism group of Ψ then the Hodge structures in $\mathbb{X}$[8] have additional structures, which can always be described as a collection of elements in some $H_{\mathbb{Z}}^{\otimes n} \otimes (H_{\mathbb{Z}}^*)^{\otimes m}$ of type $(0,0)$. Two Hodge structures in $\mathbb{X}$ are isomorphic, if they are translated by a $\gamma \in G(\mathbb{Z})$.

(2.3.2) Example (OF POLARIZED PURE HODGE STRUCTURES OF TYPE $(1,0),(0,1)$). If the weight is 1, then Ψ is symplectic and in many cases the extra structure can be chosen to consist of elements in $H_{\mathbb{Z}} \otimes H_{\mathbb{Z}}^*$ of type $(0,0)$ which are precisely endomorphisms of the Hodge structures and hence of the abelian varieties corresponding to them.
We have therefore bijections

$$\begin{aligned} \mathbb{X}/G(\mathbb{Z}) &\cong \{\text{isomorphism classes of polarized pure Hodge structures of weight 1 with additional structure}\} \\ &\cong \{\text{isomorphism classes of polarized abelian varieties with additional structure}\}. \end{aligned}$$

(2.3.3) Now suppose that there is an increasing filtration W_i on $H_{\mathbb{Q}}$ fixed. We want to construct a parameter space for mixed Hodge structures, having this particular filtration as weight filtration. For this take a group (not reductive anymore!), which we now call P and suppose that *it fixes the filtration* W_i. We denote the unipotent radical of P by W and $G := P/W$.
It is convenient to make some restrictions on $\{W_i\}_i$ and P, which we will not state (see e.g. [Pink, §2.1]). These restrictions imply, that there is a morphism $w : \mathbb{G}_m \to P$, such that w splits $\{W_i\}_i$, the unipotent radical W of P is a *central* extension

$$0 \longrightarrow U \longrightarrow W \longrightarrow V \longrightarrow 0$$

of vector groups[9] and $\mathrm{Lie}(G), \mathrm{Lie}(V), \mathrm{Lie}(U)$ are the eigenspaces of weight $0,1,2$ respectively under $Ad \circ w$ on $\mathrm{Lie}(P)$.
Elements in the unipotent radical fix $\mathrm{gr}^W(H)$ pointwise, therefore translation by elements in $W(\mathbb{C})$ also does not affect the conditions of being mixed Hodge structure. Therefore we can consider an orbit $\mathbb{X}$ under $P(\mathbb{R})W(\mathbb{C})$ of mixed Hodge structures (even $P(\mathbb{R})U(\mathbb{C})$ suffices - otherwise there is an extra stabilizer inside $W(\mathbb{C})$).
There is a corresponding orbit $\mathbb{X}_G$ under $G(\mathbb{R})$ of *pure* Hodge structures (on a subquotient of H) and a fibration $p : \mathbb{X} \to \mathbb{X}_G$. $\mathbb{X}_G$ is (as we have seen) a Hermitian symmetric space. The fibres of p are torsors under $W(\mathbb{R})U(\mathbb{C}) = W(\mathbb{C})/\mathrm{Stab}(\{F_i\}_i)$. There is a corresponding fibration $\mathbb{X}/P(\mathbb{Z}) \to \mathbb{X}_G/G(\mathbb{Z})$ whose fibres are torus[10]-torsors over abelian varieties[11].

(2.3.4) Example (OF POLARIZED MIXED HODGE STRUCTURES OF TYPE $(0,0),(1,0),(0,1),(1,1)$ AS IN (2.2)). Fix $H = U \oplus H_0 \oplus U^*$, equipped with some symplectic form on H_0 and canonical symplectic form $\langle u, u^* \rangle = -\langle u^*, u \rangle = u^*u$ on the rest. Fix the weight filtration

$$W_i(H) = \begin{cases} 0 & i \leq 0 \\ U & i = 1 \\ U^{\perp} = U^* \oplus H_0 & i = 2 \\ H & i \geq 3 \end{cases}$$

[7] under a slight additional condition on G
[8] The orbit must be chosen in a correct way
[9] algebraic groups isomorphic to $\mathbb{G}_a^n$, i.e. determined by a vector space
[10] $U(\mathbb{C})/U(\mathbb{Z})$
[11] $V(\mathbb{R})/V(\mathbb{Z})$ with an induced complex structure varying with the fibre, given by the isomorphism $V(\mathbb{R}) \cong V(\mathbb{C})/\mathrm{Stab}(\{F_i'\}_i)$, where $\{F_i'\}_i$ is any structure in the fibre

Take a polarized Hodge structure on H_0 given by some F_0^1. Then $F^1 := F_0 + U^*(\mathbb{C})$ defines a mixed Hodge structure on H! The group $\mathrm{GSp}(H_0, \mathbb{R})$ moves it, but there are 2 other possible types of moves, fixing the symplectic form *and the filtration*: For this consider $\mathrm{Sp}(H)$. Its Lie algebra is $(H \otimes H)^s$ operating by contraction with the symplectic form. Elements which preserve *and shift* the filtration at least by 1 step are of the form $(U \otimes U^\perp + U^\perp \otimes U)^s$ and there is a sequence

$$0 \longrightarrow (U \otimes U)^s \longrightarrow (U \otimes U^\perp + U^\perp \otimes U)^s \longrightarrow H_0 \otimes U \longrightarrow 0$$

The exponential (which is a polynomial on the Lie algebra of the unipotent radical!) gives the middle term (=: W) a non commutative group structure, such that the sequence is a central extension. Elements in $(U \otimes U)^s$ shift the weight filtration by 2 and elements in $H_0 \otimes U$ by 1. The corresponding subgroup of $\mathrm{GSp}(H)$ generated by this unipotent group and $\mathrm{GSp}(H_0)$ is denoted as $\mathrm{USp}(H_0, U)$ and is a semidirect product $W \rtimes \mathrm{GSp}(H_0)$.
Since W fixes $\mathrm{gr}^W(H)$, as in the general case, even its complex points $W(\mathbb{C})$ move the filtration in such a way that the properties of a mixed Hodge structure are preserved.
Together with the results of the previous paragraph we get (for any suitable subgroup P of the group USp constructed there) bijections

$$\begin{aligned} \mathbb{X}/P(\mathbb{Z}) &\cong \left\{ \begin{matrix} \text{isomorphism classes of polarized mixed Hodge structures} \\ \text{of type (0,0),(1,0),(0,1),(1,1) with additional structure} \end{matrix} \right\} \\ &\cong \{\text{isomorphism classes of polarized 1-motives with additional structure}\} \end{aligned}$$

3 Boundary components

So far we have seen no connection between pure and mixed Hodgestructures *on the same space*. In this section we will show, that imposing an 'artificial' weight filtration on a space of varying pure Hodge structures helps a lot in studying degenerations of them (and correspondingly of the abelian varieties).

3.1 Borel embedding

As already mentioned, the complex structure on an orbit $\mathbb{X}$ of mixed Hodge structures comes from associating to an element $\{F^i\}_i \in \mathbb{X}$ the corresponding element in $\mathbb{X}^\vee :=$ { the corresponding $P(\mathbb{C})$-orbit of all filtrations (without any Hodge-condition whatsoever) }. Since a structure is uniquely determined by this filtration, we get an embedding

$$\mathbb{X} \hookrightarrow \mathbb{X}^\vee = P(\mathbb{C})/P'(\mathbb{C})$$

here P' is the stabilizer of some filtration. If P is reductive, P' is a parabolic group, $\mathbb{X}^\vee$ is compact and called the **compact dual**, a notion coming from the theory of symmetric spaces.
In any case the embedding, also called **Borel embedding**, is *open*.

3.2 Definition of boundary components

For simplicity assume, $P = G$ is reductive and $G/Z(G)$ is simple.
Fix a connected component $\mathbb{X}^+$ of $\mathbb{X}$. The closure $\overline{\mathbb{X}^+}$ of the image of $\mathbb{X}^+$ under the Borel embedding is called the natural compactification of $\mathbb{X}^+$.

(3.2.1) Definition. The equivalence classes B on $\overline{\mathbb{X}^+}$ for some $\mathbb{X}^+$ with respect to the relation

$$x \sim y \Longleftrightarrow \exists \alpha : B_1(\mathbb{C}) \to \mathbb{X}^\vee : x, y \in \mathrm{im}(\alpha) \subset \overline{\mathbb{X}}$$

are called the **boundary components** of $\mathbb{X}$.

Let B be a boundary component. The stabilizer group

$$N^B(\mathbb{R}) := \{\gamma \in G(\mathbb{R}) \mid \gamma B = B\}$$

defines a parabolic subgroup N^F of $G_{\mathbb{R}}$[12].

(3.2.2) Definition. B is called a **rational** boundary component, if N^B is defined over $\mathbb{Q}$.

(3.2.3) Theorem. The association $B \mapsto N^B$ is a 1:1 correspondence between rational boundary components of $\mathbb{X}$ and *maximal* $\mathbb{Q}$-parabolic subgroups of G.
For each B there is a unique filtration W_n^B von $H_{\mathbb{Q}}$, a homomorphism $w^B : \mathbb{G}_m \to G$ (depends on the filtration $\{F^i\}_i$) with the properties

$$\begin{aligned} W_n^B &= \bigoplus_{i \le n} \{v \in H_{\mathbb{Q}} : w^B(x)v = x^i v\}, \text{i.e. } w^B \text{ splits } W^B. \\ N^B(\mathbb{C}) &= \{g \in G(\mathbb{C}) : gW_n^B \subseteq W_n^B\} \\ &= \{g \in G(\mathbb{C}) : \lim_{x \to 0} w^B(x) g w^B(x)^{-1} \text{ exists }\} \\ \mathrm{Ad}(C) \circ w^B &= (w^B)^{-1} \end{aligned}$$

and each $\{F^i\}_i \in \mathbb{X}$, together with $\{W_n^B\}_n$ defines a mixed Hodge structure on H.

3.3 Structure of groups and boundary map

There is a decomposition (depends on a base point $\{F^i\}_i \in \mathbb{X}$)

$$N^B = L^B \ltimes W^B$$

where W^B is the unipotent radical of N^B,

$$\begin{aligned} W^B &= \{g \in G : \lim_{x \to 0} w_F(x) g w_F(x)^{-1} = 1\} \\ L^B &= \{g \in G : \lim_{x \to 0} w_F(x) g w_F(x)^{-1} = g\} \\ &= G_l^B G_h^B \quad \text{(almost direct product of } \textbf{commuting} \text{ subgroups)}, \end{aligned}$$

where G_h^F and G_l^F are reductive and

$$\begin{aligned} Z^B(\mathbb{R}) &= \{g \in G(\mathbb{R}) : gx = x \forall x \in B\} = G_l^B \times W^B \\ \mathrm{Aut}(B)^+ &= G_h^B(\mathbb{R})^+. \end{aligned}$$

and there is a *central* extension

$$0 \longrightarrow U^B \longrightarrow W^B \longrightarrow V^B \longrightarrow 0$$

Therefore set[13]:

$$\begin{aligned} P^B &:= G_h^B W^B \subset G \\ \mathbb{X}^B &:= G_h(\mathbb{R}) W(\mathbb{C}) \mathbb{X} \end{aligned}$$

$\mathbb{X}^B$ is now an orbit of mixed Hodge structures (w.r.t. $\{W_n^B\}_n$) as in section (2.3) and we have

$$\mathbb{X} \subset \mathbb{X}^B \subset \mathbb{X}^\vee$$

We call the first inclusion the **boundary map**.
Furthermore $(\mathbb{X}^B)^+_{G_h}$ (defined above) is isomorphic to B itself!

[12] Caution: $N^F(\mathbb{C}) \neq \{\gamma \in G(\mathbb{C}) | \gamma B = B\}$
[13] It turns out, that G_l is irrelevant for the examination of and compactification along the boundary component B

We saw, that $(\mathbb{X}^B)^+$ is a torsor under $W^B(\mathbb{R})U^B(\mathbb{C})$ above $(\mathbb{X}^B)^+_{G_h} = B$. How does the image of $\mathbb{X}^+$ inside $(\mathbb{X}^B)^+$ look like?
Answer: There is a trivialisation $(\mathbb{X}^B)^+ \cong B \times V^B(\mathbb{R}) \times U^B(\mathbb{C})$ a bilinear form h_x on $V^B(\mathbb{R})$ with values in $U^B(\mathbb{C})$ varying real-analytically with $x \in B$ and a self-adjoint convex cone in $C^B \subset U^B(\mathbb{R})$, such that

$$\mathbb{X}^+ = \{(x, v, u) \in (\mathbb{X}^B)^+ \mid \Im(u) - h_x(v, v) \in C^B\}$$

A set of this form is called a **Siegel set of the 3^{rd} kind**.
The important fact is, that $\mathbb{X}^B$ has a much simpler structure than $\mathbb{X}$, especially $\mathbb{X}^B/P^B(\mathbb{Z})$ is simpler than $\mathbb{X}/G(\mathbb{Z})$, but in a neighborhood of the boundary component B they can be naturally identified!
There is a partial order on the set of rational boundary components, where $B_1 \leq B_2$ means more or less that B_1 may be considered as boundary component of B_2. There are then inclusions $\mathbb{X}^{B_2} \subseteq \mathbb{X}^{B_1}$, $P^{B_1} \subseteq P^{B_2}$, $U^{B_2} \subseteq U^{B_1}$, $V^{B_2} \subseteq V^{B_1}$ and w.r.t. the inclusion $U^{B_2} \subseteq U^{B_1}$, C^{B_2} is a rational boundary component (in an appropriate sense) of C^{B_1} (order is reversed!).

(3.3.1) Example. For the case $G = \mathrm{GSp}(H)$, where H is a symplectic vector space of dimension $2g$, i.e. the case of A_g, we have the following:
There is a 1:1 correspondence between

- maximal parabolic subgroups of G, i.e. boundary components of $\mathbb{X}$.
- isotropic subspaces of H.

Furthermore given an isotropic subspace U corresponding to B, choose a decomposition $H = U \oplus H_0 \oplus U^*$. We have then $L^B \cong \mathrm{GL}(U)\,\mathrm{GSp}(H_0)$, $G_l^B \cong \mathrm{GL}(U)$, $G_h^B \cong \mathrm{GSp}(H_0)$, $W^B \cong W$ (constructed in (2.3.4)), $U^B \cong (U \otimes U)^s$, $V^B \cong U \otimes H_0$, $P^B \cong \mathrm{USp}(U, H_0)$. In any case the cone $C^B \subset (U \otimes U)^s$ is the cone of positive definite bilinear forms on U^*.
If U is *maximal* isotropic, then $P^B \cong (U \otimes U)^s \rtimes \mathbb{G}_m$, $(\mathbb{X}^B)^+ = (U \otimes U)^s(\mathbb{C})$ and the realization as Siegel space

$$\mathbb{X} \subset \mathbb{X}^B$$

(now called of the 1^{st} kind) is after a choice of basis just the realization as symmetric matrixes with definite imaginary part, mentioned in the introduction. The quotient $\mathbb{X}^B/P^B(\mathbb{Z})$ is just (non-canonically) the torus with cocharacter-group $(U \otimes U)^s_{\mathbb{Z}}$!
If U is not maximal isotropic, then $\mathbb{X}^B/P^B(\mathbb{Z})$ is a torus-torsor for the torus with cocharacter group $(U \otimes U)^s_{\mathbb{Z}}$ over some power of the universal abelian variety over $\mathbb{X}^B_{\mathrm{GSp}(H_0)}/\,\mathrm{GSp}(H_0, \mathbb{Z})$.
We have seen, that

$$\mathbb{X} \subset \mathbb{X}^B$$

is given by imposing an atrificial weight filtration $0 \subseteq U \subseteq U^\perp \subseteq H$, and considering the F^1 as giving a mixed Hodge structre instead of a pure one. The inclusion induces a map

$$\mathbb{X}/\,\mathrm{GSp}(\mathbb{Z}) \to \mathbb{X}^B/\,\mathrm{USp}(U, H_0, \mathbb{Z})$$

where the first quotient parametrizes polarized abelian varieties and the second polarized 1-motives.
Analytically the 1-motives in the image of the above maps are the ones, such that $\alpha : \mathbb{Z}^n \to G$ is non-degenerate. The corresponding abelian variety is just $G/\alpha(\mathbb{Z}^n)$, as we have seen in (2.2)!

4 Toroidal compactification

The main idea of toroidal compactification is to use the structure of the $\mathbb{X}^B/P^B(\mathbb{Z})$ corresponding to some bondary component B as torus-torsor to compactify them as a 'relative' torus embedding. Then use the local isomorphism with $\mathbb{X}/P(\mathbb{Z})$ near B to glue the partial boundary to it. For this one has to compactify all $\mathbb{X}^B/P^B(\mathbb{Z})$ in a compatible way and to take the partial order of the boundary components into account.

4.1 Partial compactification

The datum to partially compactify $\mathbb{X}^B/P^B(\mathbb{Z})$ corresponding to B, is the choice of a rational polyhedral cone decomposition of the cone C^B inside $U^B(\mathbb{R})$. By this we mean a collection S^B of open rational polyhedral cones $\mathbb{R}_{>0}v_0 + \cdots + \mathbb{R}_{>0}v_n$ ($v_i \in U^B(\mathbb{Q})$!) such that each face of a polyhedral cone in S^B is again in S^B. Furthermore, we assume that $\left(\bigcup_{\sigma\in S^B}\sigma\right)^\circ = C^B$. Furthermore the family $\{S^B\}_B$ has to be compatible with the action of $G(\mathbb{Z})$ and with the partial order , which we don't want to make precise. In particular the decomposition S^B one some B determines automatically a decomposition for the boundary components of C^B, which are themselves $C^{B'}$'s for boundary components B' such that B is a boundary component of B'. We have seen, that the cone decomposition S^B determines a torus embedding

$$U^B(\mathbb{C})/U^B(\mathbb{Z}) \hookrightarrow \left(U^B(\mathbb{C})/U^B(\mathbb{Z})\right)_{S^B}$$

We can use the action of $U^B(\mathbb{C})/U^B(\mathbb{Z})$ on $\mathbb{X}^B/P^B(\mathbb{Z})$ (torus-torsor!) to glue

$$\mathbb{X}^B/P^B(\mathbb{Z}) \times^{U^B(\mathbb{C})/U^B(\mathbb{Z})} \left(U^B(\mathbb{C})/U^B(\mathbb{Z})\right)_{S^B}$$

which we call $\left(\mathbb{X}^B/P^B(\mathbb{Z})\right)_{S^B}$, the **partial compactification** of $\mathbb{X}^B/P^B(\mathbb{Z})$.
In other words, S^B describes a procedure of associating boundary components to the torus $U^B(\mathbb{C})/U^B(\mathbb{Z})$, and $X^B/P^B(\mathbb{Z})$ is a fibration with fibres isomorphic to $U^B(\mathbb{C})/U^B(\mathbb{Z})$. We add the corresponding boundary components in each fibre.
Since $\mathbb{X}^B$ is to big, we can restrict to $\mathbb{X}$ (locally around B they are isomorphic!) and get also a partial compactification

$$\left(\mathbb{X}/P^B(\mathbb{Z})\right)_{S^B}$$

4.2 Toroidal compactification

For each $B_1 \le B_2$, i.e. B_1 is a boundary component of B_2, we have an inclusion $U^{B_2} \subset U^{B_1}$ mapping C^{B_2} in a boundary component of C^{B_1}. One can show, that we get an etale map

$$\left(\mathbb{X}/P^{B_1}(\mathbb{Z})\right)_{S^{B_1}|_{C^{B_2}}} \to \left(\mathbb{X}/P^{B_2}(\mathbb{Z})\right)_{S^{B_2}}$$

The compactification is build by taking the union over all these partial compactifications and taking the quotient by the equivalence relation determined by these maps:

(4.2.1) Main theorem. There is a unique *compact* manifold $(\mathbb{X}/G(\mathbb{Z}))_{\{S^B\}_B}$, the **toroidal compactification** associated to $\{S^B\}_B$ which contains $\mathbb{X}/G(\mathbb{Z})$ as open dense submanifold, such that for each boundary component B there exists a map π^B rendering the following diagram commutative

$$\begin{array}{ccc} \mathbb{X}/P^B(\mathbb{Z}) & \hookrightarrow & \left(\mathbb{X}/P^B(\mathbb{Z})\right)_{S^B} \\ \downarrow & & \downarrow \pi^B \\ \mathbb{X}/G(\mathbb{Z}) & \hookrightarrow & (\mathbb{X}/G(\mathbb{Z}))_{\{S^B\}_B} \end{array}$$

and such that $(\mathbb{X}/G(\mathbb{Z}))_{\{S^B\}_B}$ is the union of the images of the various π^B.

If the polyhedral cone decomposition happens to be smooth (each polyhedral cone is generated by a basis of $U^B(\mathbb{Z})$), then $(\mathbb{X}/G(\mathbb{Z}))_{\{S^B\}_B}$ is smooth and the π^B's are etale.

4.3 Algebraicity of the boundary map

Consider again the case $G = \mathrm{GSp}$ and a boundary component B. Via their moduli description both $\mathbb{X}/G(\mathbb{Z})$, as well as $\mathbb{X}^B/P^B(\mathbb{Z})$ have a purely algebraic construction even over spec($\mathbb{Z}$). Equally the partial compactification of the $\mathbb{X}^B/P^B(\mathbb{Z})$ can be performed purely in algebraic terms (the corresponding rings and glueing data can be explicitely written down in terms of the coordinates of the cones and their relations).

The missing thing algebraically is the glueing coming analytically from the embedding $\mathbb{X} \subset \mathbb{X}^B$, which was just induced by imposing an artificial weight filtration (here of the from $0 \subseteq U \subseteq U^\perp \subseteq H$, where U corresponds to B). But we have seen that, in the moduli description, the inverse corresponds to taking the quotient $G/\alpha(\mathbb{Z}^n)$! The miracle is that, despite the fact that this is not an algebraic operation, it is possible *formally* algebraic. For this take the formal completion of $\left(\mathbb{X}^B/P^B(\mathbb{Z})\right)_{S^B}$ along some boundary stratum of the relative torus embedding. Let this locally be given by $\mathrm{spf}(R)$, where R is an I-adically complete ring. To this it corresponds a 1-motive $\alpha : (\mathbb{Z}^g)_R \to G_R$ where G is a semiabelian variety over R. Then there exists a semi-abelian variety $\widetilde{G}$ (think about $G/\operatorname{im}(\alpha)$) which mod I is isomorphic to G, but over the fraction field of R is 'more abelian'. In [FC] this construction is carried out and is used accompanied with Artin approximation to actually construct the toroidal compactification of A_g in purely algebraic terms (even over $\mathrm{spec}(\mathbb{Z})$).

5 Canonical models

We have seen, that for the algebraic model of A_g given by the moduli description the formal isomorphisms at the boundary are rational. In particular a function (or automorphic form, as we shall see later) which is defined over a number field K on the model of $(\mathbb{X}/\operatorname{GSp}(\mathbb{Z}))_{\{S^B\}_B}$ gives at least a formal rational function defined *over the same field* on the formal completion of the model of $\left(\mathbb{X}^B/P^B(\mathbb{Z})\right)_{S^B}$ along some boundary of the torus embedding. In the case, that $\mathbb{X}^B/P^B(\mathbb{Z})$ is just a torus itself (in this case: B is a point $\equiv U$ is *maximal* isotropic) a formal function, say along a boundary *point* of the compactification associated to a maximal dimensional cone σ, is just a power series in the generators of the dual cone. More precisely, let $\sigma \subset (U \otimes U)^s(\mathbb{R})$ be generated by $v_1, \cdots, v_{g(g+1)/2} \in (U \times U)^s$ and let $v_1^*, \cdots, v^*_{g(g+1)/2}$ be generators of the dual cone. Let f be a function on $\mathbb{X}$ invariant under $\operatorname{GSp}(\mathbb{Z})$. Consider the function as a function on $\mathbb{X}^B$ in a neighborhood of the boundary point B. The identification of $\mathbb{X}^B/P^B(\mathbb{Z}) = (U \otimes U)^s_{\mathbb{C}}/(U \otimes U)^s_{\mathbb{Z}}$ (non canonically) , is identified with $\operatorname{spec} \mathbb{C}[((U \otimes U)^s_{\mathbb{Z}})^*]$ via the exponential. So the power series is nothing but the Fourier expansion w. r. t. the lattice $(U \otimes U)_{\mathbb{Z}}$ on $\mathbb{X}$!
But the isomorphism $(\mathbb{X}^B)^+/P^B(\mathbb{Z}) = (U \otimes U)^s_{\mathbb{C}}/(U \otimes U)^s_{\mathbb{Z}}$ is not canonical! Even by taking the rational structure on $\mathbb{X}^B$ (from $H_{\mathbb{Q}}$) into account, changing it multiplies the coefficients in the Fourier expansion by roots of unity. This is equivalent to the question how the *model* of $\mathbb{X}^B/P^B(\mathbb{C})$ given by the moduli problem can be characterized in terms of the group theoretical description.
Even more complicated is the question, how the *model* of $\mathbb{X}/\operatorname{GSp}(\mathbb{Z})$ itself can be characterized in terms of the group theoretical description.
We will see that both questions can be uniformly answered, treating again mixed and pure case in the same way!

5.1 The notion of canonical model

The basic idea is, to use the functoriality of the parameter spaces - for each (admissible) subgroup of P there is a sub-parameter space whose (mixed) Hodge structures have additional structure (see 2.3)
If the subgroup happens to be a torus T, then the sub-parameter space consist of finitely many points and it turns out that the model of this sub-parameter space does not depend on the bigger group P and can be defined purely in terms of the torus T. So for each parameter space we call a model **canonical** if for every (admissible) subtorus the induced rational structure on the sub-parameter space is this particular one.
There are several difficulties:

i. The correct field of definition of the model depends on P and $\mathbb{X}$.

ii. It does not suffice to consider a model of the parameter space itself. Instead one has to consider a family of coverings of it, defined by **level structures**. A more general parameter spaces of this kind (or rather the projective limit of them) is called a **mixed Shimura variety**.

The first is easy to describe. We will not give any details.
The second consideration suffices to characterize a model uniquely, since there are (in some sense) enough points lying in sub-parameter spaces defined by tori (also called **special points**).

5.2 Parameter spaces attached to tori

We will sketch how the model of the parameter space attached to a torus T look like, and consider an embedding $T \hookrightarrow P$ where P, as usual, operates on $H_{\mathbb{Q}}$ and we fix a lattice $H_{\mathbb{Z}}$.
We assume for simplicity, that the class number of T w. r. t. $H_{\mathbb{Z}}$ is 1. There is then in $\mathbb{X}/G(\mathbb{Z})$ only 1 point, having additional structure defined by T. For each n there is then an embedding

$$T(\mathbb{Z}/n\mathbb{Z}) \hookrightarrow (\mathbb{X} \times P(\mathbb{Z}/n\mathbb{Z}))\,/P(\mathbb{Z})$$

and the image consist again of all points having additional structure defined by T, now in a covering of $\mathbb{X}/P(\mathbb{Z})$.
A model of the set $T(\mathbb{Z}/n\mathbb{Z})$ is given by the Galois action on it. We will only describe it for $T = \mathbb{G}_m$ and $T = R_{F/\mathbb{Q}}\mathbb{G}_m$ for an imaginary quadratic field $F = \mathbb{Q}(\sqrt{-D})$. For higher rank tori, the description is more involved.
For $T = \mathbb{G}_m$ the correct field of definition of $T(\mathbb{Z}/n\mathbb{Z}) = (\mathbb{Z}/n\mathbb{Z})^*$ is $\mathbb{Q}$ and $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on it via $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^*$!
For $T = R_{F/\mathbb{Q}}\mathbb{G}_m$, where $F = \mathbb{Q}(\sqrt{-D})$ (suppose $(n, D) = 1$) the correct field of definition of $T(\mathbb{Z}/n\mathbb{Z}) = (\mathcal{O}_F/n\mathcal{O}_F)^*$ is F. *Class field theory* provides us with a (so called) ray class field $F(n)$[14], with the property $\mathrm{Gal}(F(n)/F) \cong (\mathcal{O}_F/n\mathcal{O}_F)^*$. So we let act $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ via this isomorphism. This gives the correct model for an embedding $T \hookrightarrow \mathrm{GSp}(2)$, because by the theory of **complex multiplication** the Galois group acts on the torsion points of *the* elliptic curve E with complex multiplication by $\mathcal{O}_F$ by the above morphism. In particular the torsion points generate the ray class field $F(n)$. (And the image of $T(\mathbb{Z}/n\mathbb{Z}) \hookrightarrow \mathbb{X} \times \mathrm{GSp}(2, \mathbb{Z}/n\mathbb{Z})/\,\mathrm{GSp}(2, \mathbb{Z})$ consists of trivialisations of the torsion points on E)[15].
In particular this description implies, that at points coming from a torus, functions defined over a number field K have values in $K \cdot F$, where F is the field of definition for the model attached to the sub-parameter space. So, whether functions (or automorphic forms, see later) are defined over a number field can be checked either by evaluating it at special points or equally well by considering the Fourier expansion.

5.3 The 'correct' Fourier expansion

Consider again the case of A_g and a boundary point B.
We will show that the same method as above characterizes the correct Fourier expansion, i.e. the correct model of $\mathbb{X}^B/P^B(\mathbb{Z})$.
Recall $P^B \cong (U \otimes U)^s_{\mathbb{Q}} \rtimes \mathbb{G}_m$. Each (admissible) embedding $i : \mathbb{G}_m \hookrightarrow P^B_{\mathbb{Q}}$ determines such a splitting and also a splitting

$$H_{\mathbb{Q}} \cong U_{\mathbb{Q}} \oplus U^*_{\mathbb{Q}}$$

as an eigenspace decomposition of $\mathbb{G}_m$ acting trivially on U and by scalar multiplication on U^*.
The corresponding Hodge filtrations F^1 which lie in the sub parameter space defined by i are some translates of $U^*_{\mathbb{C}}$. There is an n dividing the index of $U_{\mathbb{Z}} \oplus U^*_{\mathbb{Z}}$ in $H_{\mathbb{Z}}$, (here $U^*_{\mathbb{Z}} = U^*_{\mathbb{Q}} \cap H_{\mathbb{Z}}$) such that there are modulo $P^B_{\mathbb{Z}}$ exactly $\varphi(n)$ points in this sub parameter space, determined by an embedding

$$i' : (\mathbb{Z}/n\mathbb{Z})^* \hookrightarrow \mathbb{X}^B/P^B_{\mathbb{Z}}.$$

(This is, however, easier to make precise if one works with adeles.)
To make this compatible with the Galois structure on $\mathbb{Z}/n\mathbb{Z}$ one has to use a trivialization (gives a model)

$$\mathbb{X}^B/P^B_{\mathbb{Z}} \cong (U \otimes U)^s_{\mathbb{C}}/(U \otimes U)^s_{\mathbb{Z}} = \mathbb{G}_m \otimes (U \otimes U)^s_{\mathbb{Z}}$$

determined by a splitting $H_{\mathbb{Z}} = U_{\mathbb{Z}} \oplus U^*_{\mathbb{Z}}$ (hence an $F^1 = U^*_{\mathbb{C}}$) defined over $\mathbb{Z}$! On this gives the correct model and the correct Fourier expansion.
One checks that in this case *all* morphisms i' above are defined over $\mathbb{Q}$!

[14] see this as generalization of $\mathbb{Q}(\zeta_n)$!
[15] If the class number is not 1 then things are a bit more involved and one better works with adeles.

6 Automorphic forms

coming soon...?

6.1 Definition of automorphic form

6.2 The standard principal bundle

6.3 Extensions to the boundary

6.4 Examples