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Lectures on Shimura Varieties

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1 Preamble

The group $\mathrm{SL}_2(\mathbb{R})$ acts on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ by fractional linear transformations, yielding an isomorphism $\mathrm{Aut}(\mathbb{H}) \cong \mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$. Let Γ be a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ defined by congruence conditions. The quotients

$$\Gamma \backslash \mathbb{H},$$

more precisely, certain disjoint unions X of them, are the easiest examples of Shimura varieties. The following observations can be made:

1. X is an algebraic variety, *canonically* defined over $\mathbb{Q}$ (sometimes even over $\mathbb{Z}$). It is the solution to a moduli problem for elliptic curves with level structures.

2. There are canonical line bundles on X, whose sections are precisely the modular forms.

3. X can be *naturally* compactified.

4. There are distinguished points on X, the so called *CM-points*[1] or *special points*, where the Galois action can be described explicitly in terms of class field theory. There are sufficiently many of those to characterize the $\mathbb{Q}$-model of X uniquely.

5. There is a huge ring of correspondences, the so called Hecke algebra, acting on X and the modular forms. It can be used to reveal a deep connection between modular forms and 2-dimensional Galois representations.

All these facts have analogues for $\mathbb{H}$ replaced by an arbitrary *Hermitian symmetric domain* $\mathbb{D}$ and for Γ replaced by a subgroup of $G(\mathbb{Z})$ defined by congruence conditions. Here G is a linear algebraic group defined over $\mathbb{Z}$ (semi-simple over $\mathbb{Q}$) with a surjective homomorphism with finite kernel

$$G(\mathbb{R}) \to \mathrm{Aut}(\mathbb{D})^+.$$

The associated Shimura variety is then roughly

$$\Gamma \backslash \mathbb{D}.$$

It is a fundamental theorem that this is in fact always an algebraic variety defined over a number field.

2 Introduction (example of modular curves)

2.1 Elliptic curves

The easiest Shimura varieties are the **modular curves** which are algebraic varieties of dimension 1 parametrizing elliptic curves. Recall that an elliptic curve (defined over a field K of characteristic 0) can be given as a curve $E \subset \mathbb{P}^2$ with affine equation (Weierstrass equation):

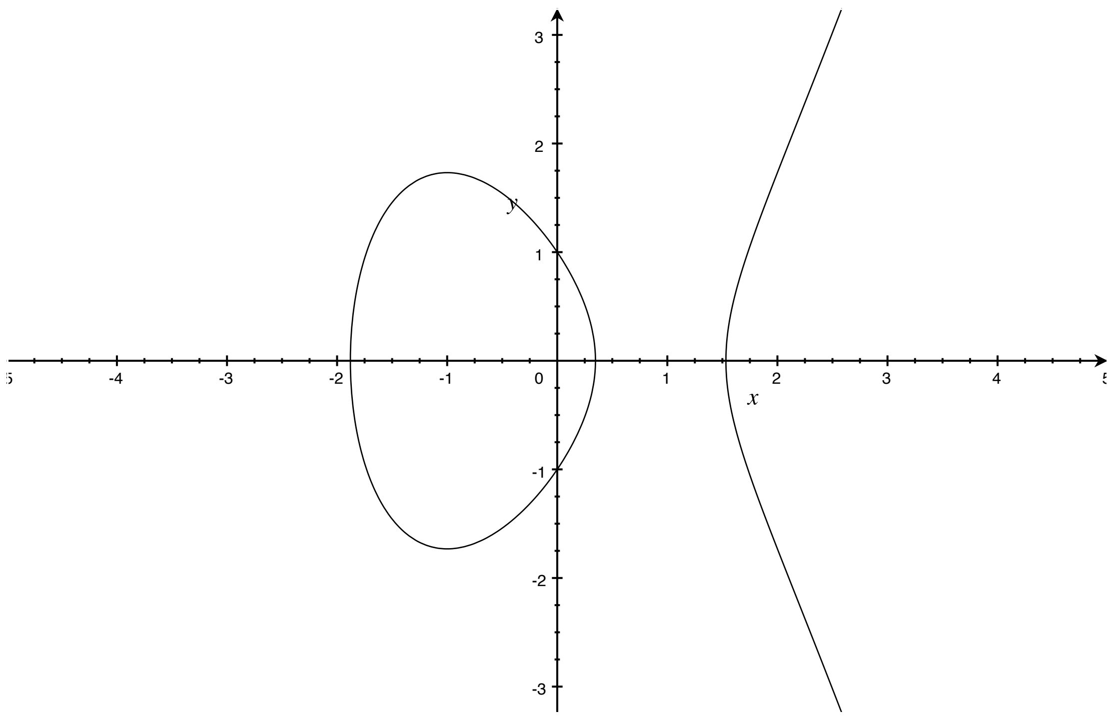
$$y^2 = x^3 + px + q, \tag{1}$$

[1] CM stands for 'complex multiplication' and means additional endomorphisms of an elliptic curve or Abelian variety.

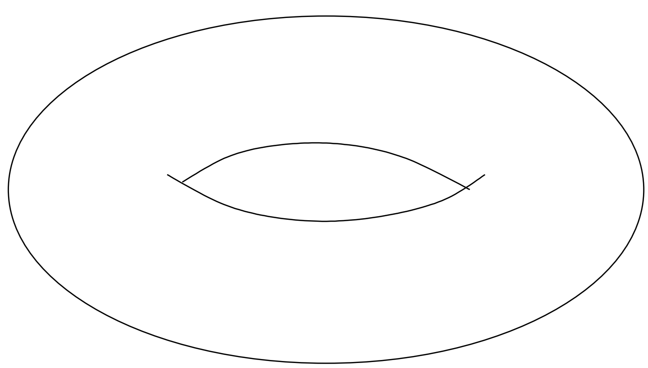
where p, q are elements of K [2]. During the introduction K will always be a subfield of $\mathbb{C}$. For each field L containing K we define the set of L-valued points of E as

$$E(L) := \mathrm{Hom}(\mathrm{spec}(L), E) = \{(x,y) \in L^2 \mid y^2 = x^3 + px + q\} \cup \{\infty\}.$$

A fundamental fact about elliptic curves is that $E(L)$ is an Abelian group for all fields $L \supset K$. For example if $K \subset \mathbb{R}$, the affine part of $E(\mathbb{R})$ looks like the following curve:



and $E(\mathbb{C})$ has the structure of a complex torus:



In other words, there is a 2-dimensional lattice $L \subset \mathbb{C}$ such that $E(\mathbb{C}) \cong \mathbb{C}/L$ as analytic manifolds. In fact we have

Theorem 2.1. *There is a bijection*

$$\{\textit{elliptic curves defined over } \mathbb{C}\}_{/Iso.} \cong \{\textit{rank 2 lattices } L \subset \mathbb{C}\}_{/\sim}.$$

Here for two lattices $L \sim L'$ if and only if there exists an $\alpha \in \mathbb{C}^*$ such that $L = \alpha L'$.

[2] Equivalently: an elliptic curve, defined over K, is a smooth projective curve of genus 1 over $\mathrm{spec}(K)$ together with a distingushed K-rational point.

Proof (Sketch). With a rank 2 lattice $L \subset \mathbb{C}$ we can associate the complex manifold $\mathbb{C}/L$ and there is an explicit biholomorphic map

$$\begin{array}{rcl} \mathbb{C}/L & \to & E_L(\mathbb{C}) = \{(x,y) \in \mathbb{C}^2 \mid y^2 = x^3 + px + q\} \cup \{\infty\} \\ z & \mapsto & (\wp(z), \frac{1}{2}\wp'(z)) \end{array}$$

where $\wp$ is the Weierstrass $\wp$-function associated with L:

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in L \setminus \{0\}} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}$$

and we have

$$p = -15 \sum_{\lambda \in L \setminus \{0\}} \frac{1}{\lambda^4}, \qquad q = -35 \sum_{\lambda \in L \setminus \{0\}} \frac{1}{\lambda^6}.$$

Conversely, let E be an elliptic curve defined over $\mathbb{C}$. Then one can show that $E(\mathbb{C})$ is a commutative complex Lie group which is connected and compact. Since it is a commutative complex Lie group, the exponential map

$$\exp : \mathbb{C} \cong T_e(E(\mathbb{C})) \to E(\mathbb{C})$$

is an analytic group homomorphism with the property that a neighborhood of 0 is mapped biholomorphically onto a neighborhood of $e = \infty$. Hence the image has to be an open subgroup and therefore also closed. Therefore the map is surjective because $E(\mathbb{C})$ is connected. Its kernel is a discrete subgroup, hence a lattice $L \subset \mathbb{C}$. If it would not have rank 2 the quotient $E(\mathbb{C}) \cong \mathbb{C}/L$ would not be compact. □

Remark 2.2. *The theorem should be formulated more accurately by saying: There is an equivalence of categories*

$$\{\textit{elliptic curves defined over } \mathbb{C}\} \xrightarrow{\sim} \{\textit{rank 2 lattices } L \subset \mathbb{C}\}$$

where the right hand side has morphism sets

$$\mathrm{Hom}(L, L') = \{\alpha \in \mathbb{C}^* \mid \alpha L \subset L'\}.$$

2.2 Analytic parameter space

Using Theorem 2.1, the analytic classification of elliptic curves defined over $\mathbb{C}$ is easy. We have

Proposition 2.3. *There is a bijection:*

$$\{\textit{rank 2 lattices } L \subset \mathbb{C}\}_{/\sim} \cong \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}.$$

Here $\mathrm{SL}_2(\mathbb{Z})$ *(or even* $\mathrm{SL}_2(\mathbb{R})$*) acts on* $\mathbb{H}$ *via fractional linear transformations*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \tau = \frac{a\tau + b}{c\tau + d}.$$

Proof. A complex number $\tau \in \mathbb{H}$ is sent to the lattice $\mathbb{Z} + \tau\mathbb{Z} \subset \mathbb{C}$, and a lattice $\alpha\mathbb{Z} + \beta\mathbb{Z} \subset \mathbb{C}$ (where the basis is chosen compatible with the standard orientation of $\mathbb{C}$) is sent to $\frac{\beta}{\alpha} \in \mathbb{H}$. A different oriented basis α', β' differs by a transformation $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, i.e. $\beta' = a\beta + b\alpha$ and $\alpha' = c\beta + d\alpha$. It is sent to $\frac{\beta'}{\alpha'} = \frac{a\beta + b\alpha}{c\beta + d\alpha} = \frac{a\frac{\beta}{\alpha} + b}{c\frac{\beta}{\alpha} + d}$, hence to the corresponding $\mathrm{SL}_2(\mathbb{Z})$-translate of $\frac{\beta}{\alpha}$. □

We may try to extend the classification to families, which would have the advantage that also the complex structure on the parameter space is determined uniquely by this. Given an analytic manifold (or even analytic space) V, by a **family of elliptic curves over** V we intend a proper holomorphic submersion

$$p : E \to V$$

whose fibers are complex tori of (complex) dimension 1. Then we may consider the moduli problem, or moduli functor:

$$F(V) := \{ \text{ families of elliptic curves over } V \ \}_{/\mathrm{Iso.}}$$

and ask whether it is representable, i.e. whether there is an analytic manifold M together with isomorphisms

$$F(V) \cong \mathrm{Hom}(V, M) \tag{2}$$

which are functorial in V. This would imply that M is uniquely determined (up to unique isomorphism) together with a universal family of elliptic curves $E_{\mathrm{univ}} \to M$ which would correspond to id_M on the right hand side of (2). However, on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ such a universal family does not exist. One reason is that such a space could not distinguish between different *isotrivial*[3] elliptic curves over a base V with $\pi_1(V) \neq \{1\}$. Isotrivial elliptic curves exist because elliptic curves have non-trivial automorphisms. Another fact in which this problem is reflected is the following. Consider the canonical family of complex tori which *does* exist over $\mathbb{H}$:

$$\mathbb{Z}^2\backslash(\mathbb{H} \times \mathbb{C}) \to \mathbb{H}.$$

Here $\mathbb{Z}^2$ acts by $\begin{pmatrix} a \\ b \end{pmatrix} (\tau, z) = (\tau, z + a + b\tau)$. This means: the fiber over $\tau \in \mathbb{H}$ is the elliptic curve $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. On this canonical family $\mathrm{SL}_2(\mathbb{Z})$ acts by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, (c\tau + d)^{-1} z \right).$$

When we divide out this action, however, the elliptic curves in the fibers get divided by their automorphism group. The only non-trivial automorphism is for almost all τ the multiplication by -1 induced by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, and for those τ which are in the orbit of i or $\omega = \exp(\frac{2\pi i}{6})$ there are more.
These problems get solved by introducing level structures. We will explain this now in the algebraic context.

2.3 Algebraic parameter space

Recall that for an elliptic curve with affine equation $y^2 = x^3 + px + q$, the j-invariant is defined as

$$j(E) = j(p, q) = 1728 \frac{(4p)^3}{\Delta(p, q)},$$

where $\Delta(p, q) = -16(4p^3 + 27q^2)$ is the discriminant of the polynomial $x^3 + px + q$.

Theorem 2.4. *For any algebraically closed field K, j induces a bijection*

$$\{ \text{ elliptic curves defined over } K \ \}_{/Iso.} \cong K.$$

Remark 2.5. *In particular for $K = \mathbb{C}$, we get a bijection $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \to \mathbb{C}$ which is even holomorphic and induced by the famous analytic j-function*

$$j(\tau) = \frac{1}{q} + 744 + 196884q + \cdots$$

with $q = \exp(2\pi i\tau)$.

Trying to extend the parametrization of Theorem 2.4 to families, we run into similar problems as in the analytic picture: First for a variety V (or even any scheme) we define a **family of elliptic curves** over V as a smooth morphism of varieties

$$p : E \to V$$

such that $p^{-1}(v)$ is an elliptic curve for all $v \in V$ (and which has a section $s : V \to E$).

[3]'Isotrivial' means that all fibers are isomorphic to each other.

For example, the Weierstrass equation (1) for elliptic curves gives a family of elliptic curves over $\mathbb{A}^2 \setminus V(\Delta(p,q))$

$$E_{p,q} \to \mathbb{A}^2 \setminus V(\Delta(p,q)).$$

Again we consider the functor (moduli problem):

$$F(V) = \{ \text{ families of elliptic curves over } V \}_{/\text{Iso.}}.$$

For similar reasons as in the analytic picture F is not representable. For example there is no Cartesian square

$$\begin{tikzcd} E_{p,q} \arrow[r] \arrow[d] & E_{\text{univ}} \arrow[d] \\ \mathbb{A}^2 \setminus V(\Delta(p,q)) \arrow[r, "j"] & \mathbb{A}^1 \end{tikzcd}$$

although j is a well-defined morphism of varieties.
This is remedied by eliminating the automorphisms of the objects in the moduli problem. Instead of considering bare elliptic curve, we equip them with a level-N-structure (for any fixed integer N). The non-representability is not the only reason; the resulting moduli spaces are also arithmetically much more interesting. A level-N-structure on an elliptic curve, or more generally on a family of elliptic curves $E \to V$ is a rigidification of the N-torsion points, i.e. an isomorphism of group varieties:

$$(\mathbb{Z}/N\mathbb{Z})^2_V \cong E[N]$$

Here $E[N]$ for a single elliptic curve E is the variety of points P on E such that

$$N \cdot P := \underbrace{P + \cdots + P}_{N \text{ times}} = 0.$$

Even though those points may not be defined over K, if E is defined over K, the 0-dimensional variety $E[N]$ is. This is because the group structure on E is defined by polynomials with coefficients in K, hence the equation $N \cdot P = 0$ is described by polynomials with coefficients in K. Over $\mathbb{C}$, writing $E(\mathbb{C}) = \mathbb{C}/L$, we immediately see that $E[N](\mathbb{C}) = \frac{1}{N}L/L \cong (\mathbb{Z}/N\mathbb{Z})^2$. This holds true more generally:

Theorem 2.6. *For an arbitrary algebraically closed field K of characteristic 0 and an elliptic curve E, defined over K, we have a group isomorphism*

$$E[N](K) \cong (\mathbb{Z}/N\mathbb{Z})^2.$$

For a family of elliptic curves $E \to V$, the variety $E[N]$ is obtained by taking the N-torsion points in each fiber. It is a group variety over V with 0-dimensional fibers. $(\mathbb{Z}/N\mathbb{Z})^2_V$ is the constant group variety over V with group $(\mathbb{Z}/N\mathbb{Z})^2$.
This definitions allow us to modify the moduli problem F and define

$$F_N(V) = \left\{ \begin{array}{c} \text{families of elliptic curves } E \to V \\ \text{with a level-}N\text{-structure } \xi : (\mathbb{Z}/N\mathbb{Z})^2_V \cong E[N] \end{array} \right\}_{/\text{Iso.}}$$

Now we have:

Theorem 2.7. *If $N > 3$, F_N is representable by a smooth curve M_N defined over $\mathbb{Q}$.*

Note that this implies that M_N is uniquely determined up to a unique isomorphism. Note also that F_N and hence M_N carry a natural action of the group $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ *on the right*.
These M_N are called **modular curves**. Also analytically one can introduce level-structures in the same way and obtains a $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$-equivariant isomorphism of Riemann surfaces

$$M_N(\mathbb{C}) \cong \mathrm{GL}_2(\mathbb{Z})\backslash(\mathbb{H}^{\pm} \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})). \tag{3}$$

Here we switched from SL_2 to GL_2 because otherwise we would have to explain what 'determinant 1', i.e. 'symplectic', means for a level-N-structure. This can be done using the Weil pairing, but is somewhat unnatural anyway (it would force the natural field of definition of M_N to be $\mathbb{Q}(\zeta_N)$). The right hand side of (3) can be rewritten as

$$\mathrm{GL}_2(\mathbb{Z})\backslash\mathbb{H}^{\pm} \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) = \bigcup_{\alpha\in(\mathbb{Z}/N\mathbb{Z})^*} \Gamma(N)\backslash\mathbb{H}$$

where

$$\Gamma(N) = \{X \in \mathrm{SL}_2(\mathbb{Z}) \mid X \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N\}.$$

Note that the isomorphism (3) does not characterize the structure of M_N as a variety over $\mathbb{Q}$! To understand the problem consider the following two curves over $\mathbb{R}$:

$$X^2 + Y^2 = 1 \qquad X^2 - Y^2 = 1$$

They are certainly not isomorphic over $\mathbb{R}$ (for example the $\mathbb{R}$-points of the first are connected whereas those of the second are not) but become isomorphic over $\mathbb{C}$. For the modular curves Theorem 2.7 characterized a meaningful $\mathbb{Q}$-rational model. When we generalize the picture to higher dimensions, analogous moduli problems do not exist anymore (or, at least, only conjecturally). Therefore we have to find another characterization for the 'right' models, defined over a number field. We will now discuss, how the M_N's can be characterized without using the moduli problem.

2.4 Characterization of the $\mathbb{Q}$-rational structure of M_N

Recall that for two elliptic curves over $\mathbb{C}$, we have

$$\mathrm{Hom}(\mathbb{C}/L, \mathbb{C}/L') = \{\alpha \in \mathbb{C} \mid \alpha L \subseteq L'\}.$$

From this follows that there are two possibilities for the ring of endomorphisms

$$\mathrm{End}(\mathbb{C}/L) = \begin{cases} \mathbb{Z} \\ R \subseteq \mathcal{O}_K \quad K = \mathbb{Q}(\sqrt{-D}) \end{cases}$$

In the second case R is an order in the ring of integers $\mathcal{O}_K$, and we say E has **complex multiplication (CM)** by R.

Example 2.8. *Let $\mathfrak{a} \subset \mathcal{O}_K$ be an ideal. Then*

$$E_{\mathfrak{a}} := \mathbb{C}/\mathfrak{a}$$

is an elliptic curve with CM by $\mathcal{O}_K$.

In fact, we have the following:

Theorem 2.9. *The function $\mathfrak{a} \mapsto E_{\mathfrak{a}}$ induces a bijection*

$$\mathrm{Cl}(K) \cong \{ \text{ elliptic curves over } \mathbb{C} \text{ with CM by } \mathcal{O}_K \ \}_{/Iso.},$$

where $\mathrm{Cl}(K)$ is the class group of K.

Since the endomorphisms of elliptic curves are morphisms of algebraic varieties, all Galois conjugates of an elliptic curve with CM by $\mathcal{O}_K$ have also CM by $\mathcal{O}_K$. The theorem implies therefore (together with the representability of some M_N) that all those elliptic curves are defined over number fields. For level $N = 1$ we get an embedding $\iota_1 : \mathrm{Cl}(K) \hookrightarrow M_1(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}}$. In fact, we can also describe (part of) the preimage of $\mathrm{Cl}(K)$ under the projections from the various M_N to M_1:

$$\begin{tikzcd} \mathrm{Cl}_N(K) \arrow[r, hook, "\iota_N"] \arrow[d, two heads] & M_N(\overline{\mathbb{Q}}) \arrow[d] \\ \mathrm{Cl}(K) \arrow[r, hook, "\iota_1"] & M_1(\overline{\mathbb{Q}}) \end{tikzcd}$$

Here $\mathrm{Cl}_N(K)$ is an Abelian group which sits naturally in an exact sequence

$$1 \longrightarrow (\mathcal{O}_K/N\mathcal{O}_K)^* \longrightarrow \mathrm{Cl}_N(K) \longrightarrow \mathrm{Cl}(K) \longrightarrow 1$$

We will not give its definition here, it can best be described adelically. Class field theory states that there is a natural field extension K_N/K, such that there is a natural isomorphism

$$\mathrm{rec} : \mathrm{Cl}_N(K) \cong \mathrm{Gal}(K_N/K).$$

K_N is called the **ray class field of level** N. The main theorem of complex multiplication, which we will discuss later in broader generality, implies that the points in the image of ι_N are defined over K_N, and the Galois action is given simply by

$$^{\sigma}\iota_N([\mathfrak{a}]) = \iota_N(\mathrm{rec}^{-1}(\sigma) \cdot [\mathfrak{a}]) \tag{4}$$

for all $\sigma \in \mathrm{Gal}(K_N/K)$.
The importance of the theory of complex multiplication for the characterization of models of Shimura varieties comes from the following:

Theorem 2.10. *The system $\{M_N\}_N$ is the unique system of $\mathbb{Q}$-rational algebraic models of the Riemann surfaces $\{\mathrm{GL}_2(\mathbb{Z})\backslash(\mathbb{H} \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}))\}_N$ that satisfy (4) for all imaginary quadratic fields K.*

A condition like (4) will be formulated for all Shimura varieties, and will again uniquely characterize a system of models. However, without a modular interpretation, it is a different and more difficult question, whether such a system does exist or not. The possibility of characterizing meaningful arithmetic models of locally symmetric varieties this way had been discovered by Shimura.

2.5 The Hermitian symmetric domain $\mathbb{H}$ and generalizations

The upper half plane $\mathbb{H}$ has the following properties:

1. We have a isomorphism of real manifolds

$$\begin{array}{ccc} \mathbb{H} & \overset{\sim}{\longleftarrow} & \mathrm{SL}_2(\mathbb{R})/K \\ \dfrac{ai+b}{ci+d} & \longleftarrow\!\shortmid & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array}$$

   where $K = \mathrm{Stab}(i)$ the following maximal compact subgroup of $\mathrm{SL}_2(\mathbb{R})$:

$$K = U_1 = \{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 = 1\}.$$

2. For each $\tau \in \mathbb{H}$, there is a morphism $u_\tau : U_1 \to \mathrm{Aut}(\mathbb{H}) = \mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$ fixing τ and such that $z \mapsto u_\tau(z)$ defines the complex structure on the tangent space $T_\tau(\mathbb{H})$.

3. $\mathbb{H}$ carries a $\mathrm{SL}_2(\mathbb{R})$-invariant Hermitian metric w.r.t. the complex structure on $\mathbb{H}$.

Actually 3. follows from 1. and 2. because K is a compact group, and so by averaging over K any Hermitian metric on $T_i(\mathbb{H})$, and then translating, we produce an invariant Hermitian metric. The morphism u_i from 2. is not the obvious one (that one acts by multiplication with z^2 on the tangent space) but its square root.
The generalization of this is the definition of a **Hermitian symmetric domain** $\mathbb{D}$:

1. There is an isomorphism of real manifolds

$$\mathbb{D} \overset{\sim}{\longleftarrow} G/K$$

   where G is a real semi-simple Lie group and K is a maximal compact subgroup of G.

2. For each $\tau \in \mathbb{D}$, there is a morphism $u_\tau : U_1 \to \mathrm{Aut}(\mathbb{D}) = G^{\mathrm{ad}}$ fixing τ and such that $z \mapsto u_\tau(z)$ defines a complex structure on the tangent space $T_\tau(\mathbb{D})$.

3. $\mathbb{D}$ carries a G-invariant Hermitian metric w.r.t. the (almost) complex structure from 2. on $\mathbb{D}$.

Again 3. follows from 1. and 2. It also follows that $\mathbb{D}$ is an honest complex manifold (i.e. the almost complex structure is integrable). We will see other, equivalent ways of defining a Hermitian symmetric domain in the course of this lecture.

Example 2.11.

$$\mathbb{H}_g = \{X \in M_{g\times g}(\mathbb{C}) \mid {}^tX = X,\ \mathrm{Im}(X) \textit{ positive definite } \}$$

is called **Siegel's upper half space** *of genus (dimension) g and we have*

$$\mathbb{H}_g = \mathrm{Sp}_{2g}(\mathbb{R})/U_g,$$

where Sp_{2g} *is the symplectic group in dimension* $2g$ *and* U_g *is the unitary group in complex dimension* g.

The questions are

1. Do appropriate quotients $\Gamma\backslash\mathbb{D}$ have a model over $\mathbb{Q}$ (or at least over a number field)?

2. How can these models be characterized?

The group Γ in Question 1. will be any **congruence subgroup**, i.e. a subgroup defined by congruence conditions in the group of integral matrices in a realization of G as a matrix group defined over $\mathbb{Q}$. It is not easy at all to see that $\Gamma\backslash\mathbb{D}$ has an algebraic model even only over $\mathbb{C}$. This is, however, true by the famous Theorem of Baily-Borel that we will discuss during the lecture. Those algebraic models are the **Shimura varieties**. We will also discuss how one can obtain their sought-for models over number fields in many cases. Question 2. will be answered by a generalization of Theorem 2.10. A modular description, in general, exists only conjecturally in terms of motives. We will discuss it briefly in the end of the lecture.

2.6 The Langlands program

For every $N'|N$ we have a projection map $M_N \to M_{N'}$ and in addition every X_N carries an action of the group $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. We will see that on the projective limit of this system, we have even an action of $\mathrm{GL}_2(\mathbb{A}_f)$, where $\mathbb{A}_f$ is the ring of finite adeles of $\mathbb{Q}$, hence in particular an action of $\mathrm{GL}_2(\mathbb{Q})$. Those actions are not apparent at all on the individual M_N. On those we get at most an action by correspondences, i.e. multi-valued maps. These correspondences are called Hecke correspondences and are the more classical description.

Example 2.12. *Consider a prime p and the group*

$$\Gamma_0(p) = \{X \in \mathrm{SL}_2(\mathbb{Z}) \mid X \equiv \begin{pmatrix} * & * \\ & * \end{pmatrix} \pmod{p}\}.$$

We have the equation

$$\begin{pmatrix} p & \\ & 1 \end{pmatrix} \Gamma_0(p) \begin{pmatrix} p^{-1} & \\ & 1 \end{pmatrix} \subset \mathrm{SL}_2(\mathbb{Z})$$

hence the multiplication by the matrix $\begin{pmatrix} p & \\ & 1 \end{pmatrix}$ *induces a well-defined morphism* $\Gamma_0(p)\backslash\mathbb{H} \to \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. *Together with projection induced by the identity, we get the diagram*

$$\begin{tikzcd} & \Gamma_0(p)\backslash\mathbb{H} \arrow[dl] \arrow[dr, "\begin{pmatrix} p & \\ & 1 \end{pmatrix}"] & \\ \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} & & \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \end{tikzcd}$$

This diagram has the following modular interpretation:

$$\begin{tikzcd} & \left\{ \begin{array}{c} \rho : E \to E' \text{ Isogeny of} \\ \text{elliptic curves over } \mathbb{C} \text{ of degree } p \end{array} \right\}_{/\ Iso.} \arrow[dl, "\rho\mapsto E"'] \arrow[dr, "\rho\mapsto E'"] & \\ \{\text{elliptic curves over } \mathbb{C}\}_{/\ Iso.} & & \{\text{elliptic curves over } \mathbb{C}\}_{/\ Iso.} \end{tikzcd}$$

Here an isogeny $\rho : E \to E'$ *of degree* p *of elliptic curves is a homomorphism with finite kernel of cardinality* p. *The correspondence (or multi-valued map) represented by this diagram is called the* **Hecke correspondence** T_p.

The algebra (of correspondences of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$) generated by the T_p is actually commutative

$$\mathcal{H} = \mathbb{C}[T_2, T_3, T_5, \dots]$$

and called the **Hecke algebra**.
Let q be a prime. Also the Riemann surface $\Gamma_0(q)\backslash\mathbb{H}$ has a rational model

$$M_{0,q} := M_q / \begin{pmatrix} * & * \\ & * \end{pmatrix}$$

and a similar algebra $\mathcal{H}_q$ acts on it by correspondences (basically only the operator T_q has to be changed).
The model $M_{0,q}$ is a smooth curve which can be compactified in a unique way to a projective curve:

$$M_{0,q} \hookrightarrow \overline{M_{0,q}},$$

which is still defined over $\mathbb{Q}$. The complement is a zero dimensional variety, i.e. over the algebraic closure of $\mathbb{Q}$ it contains a finite number of points, the **cusps** of the curve.
Consider the cohomology group

$$H^1(\overline{M_{0,q}}(\mathbb{C}), \mathbb{Z}).$$

By tensoring with $\mathbb{Q}, \mathbb{C}$ and $\mathbb{Z}/N\mathbb{Z}$, respectively, we get

$$\begin{aligned} H_B &:= H^1(\overline{M_{0,q}}(\mathbb{C}), \mathbb{Q}), \\ H_{dR} &:= H^1(\overline{M_{0,q}}(\mathbb{C}), \mathbb{C}), \\ H_N &:= H^1(\overline{M_{0,q}}(\mathbb{C}), \mathbb{Z}/N\mathbb{Z}). \end{aligned}$$

These all carry a compatible action of the Hecke algebra which is selfadjoint w.r.t. a certain scalar product, hence all three spaces decompose into a direct sum of irreducible modules for the Hecke algebra:

$$\begin{aligned} H_B &:= \bigoplus_i H_B^{(i)} \\ H_{dR} &:= \bigoplus_i H_{dR}^{(i)} \\ H_N &:= \bigoplus_i H_N^{(i)} \end{aligned}$$

The important point is, that H_{dR} and H_N carry additional structures, the first being of "modular nature" tied to the representation theory of SL_2 and the second of "arithmetic nature".

1. We have a decomposition, which is compatible with the action of the Hecke algebra:

$$H_{dR} = S_2 \oplus \overline{S_2}$$

where

$$S_2 = H^0(\overline{M_{0,q}}(\mathbb{C}), \Omega^1).$$

This is equivalently the space of holomorphic modular forms of weight 2 which vanish at all cusps (cusp forms), i.e. functions

$$f : \mathbb{H} \to \mathbb{C}$$

which transform according to

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 f(\tau)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that in the Fourier expansion

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n \qquad q(\tau) = \exp(2\pi i \tau)$$

the first coefficient a_0 vanishes (and similarly for the other cusps $\neq \infty$).

2. The space H_N carries an (Hecke invariant) action of the Galois group. Note that we have

$$H^1(\overline{M_{0,q}}(\mathbb{C}), \mathbb{Z}/N\mathbb{Z}) = \left\{ \begin{array}{c} \text{Galois covers } X \to \overline{M_{0,q}}(\mathbb{C}) \text{ together with} \\ \text{an injective homomorphism } \mathrm{Gal}(X/\overline{M_{0,q}}(\mathbb{C})) \hookrightarrow \mathbb{Z}/N\mathbb{Z} \end{array} \right\}$$

Let $X \to \overline{M_{0,q,\mathbb{C}}}$ be such a Galois cover, considered as a morphism of algebraic varieties. For each automorphism $\sigma \in \mathrm{Aut}(\mathbb{C})$ the morphism ${}^{\sigma}X \to {}^{\sigma}\overline{M_{0,q,\mathbb{C}}} = \overline{M_{0,q,\mathbb{C}}}$ is again a Galois cover and σ induces an isomorphism

$$\sigma : \mathrm{Gal}(X : \overline{M_{0,q}}(\mathbb{C})) \cong \mathrm{Gal}({}^{\sigma}X : \overline{M_{0,q}}(\mathbb{C})).$$

This defines an action of $\mathrm{Aut}(\mathbb{C})$ on this group. Alternatively one can see $H^1(\overline{M_{0,q}}(\mathbb{C}), \mathbb{Z}/N\mathbb{Z})$ as the first etale cohomology group of $\overline{M_{0,q,\mathbb{C}}}$ with values in $\mathbb{Z}/N\mathbb{Z}$ or as (the dual of) the group of N-torsion points of the Jacobian of $\overline{M_{0,q}}$.

Because these structures are Hecke equivariant, we have similar structures on the $H^{(i)}$.
The easiest case occurs when $H^{(i)}_{dR}$ has dimension 2. In this case one can show that we have

$$\begin{aligned} H^{(i)}_B &:= H^1(E(\mathbb{C}), \mathbb{Q}) \\ H^{(i)}_{dR} &:= H^1(E(\mathbb{C}), \mathbb{C}) = S^{(i)}_2 \oplus \overline{S^{(i)}_2} \\ H^{(i)}_N &:= H^1(E(\mathbb{C}), \mathbb{Z}/N\mathbb{Z}) = E[N]^* \end{aligned}$$

for an elliptic curve E, defined over $\mathbb{Q}$. Elliptic curves occurring this way are called **modular**. In this case $S^{(i)}_2$ is one dimensional, and hence it is generated by a cusp form

$$f_E(\tau) = \sum_{n \geq 1} a_n q^n \qquad q = \exp(2\pi i \tau)$$

which is an eigenform of all the Hecke operators. One can show that if f is normalized, i.e. $a_1 = 1$, then a_p *is* the eigenvalue of T_p and is an integer for all primes p.
For $p \neq q$ there exists a model of E, defined over $\mathbb{Z}$, such that $E_{\mathbb{F}_p}$ is again an elliptic curve. By analyzing the action of T_p on this reduction, one obtains the surprising interpretation (Eicher-Shimura congruence relation):

$$\mathrm{tr}(\sigma_p | H^{(i)}_N) \equiv a_p$$

where σ_p is the inverse of any Frobenius element of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ at p. By the Lefschetz fixed-point-formula (which in the case of elliptic curves is elementary) we therefore have:

$$\#E(\mathbb{F}_p) = 1 + p - a_p.$$

From this one can deduce that the L-series of E can be written as

$$L(E,s) = L_q(E,s) \prod_{p \neq q} \frac{1}{1 - a_p p^{-s} + p^{-2s+1}}$$

and coincides with the L-series of f

$$L(f,s) = \sum_n \frac{a_n}{n^s},$$

which satisfies the functional equation $\Lambda(f,s) = \pm\Lambda(f, 2-s)$ where

$$\Lambda(f,s) = q^{\frac{s}{2}}(2\pi)^{-s}\Gamma(s)L(E,s)$$

is the complete L-series of f. One does not know of a different proof of the functional equation for $L(E,s)$, which does not use the modularity of E.

Example 2.13. *For example for $q = 11$, the curve $\overline{M_{0,q}}$ is itself an elliptic curve E with equation*

$$E : y^2 = -44x^3 + 56x^2 - 20x + 1$$

and the associated Hecke eigenform can be described as

$$f_E(\tau) = \eta^2(\tau)\eta^2(11\tau),$$

where

$$\eta(\tau) = \exp(\frac{2\pi i}{24}) \prod_{n=1}^{\infty}(1 - q^n) \qquad q = \exp(2\pi i \tau).$$

In the end we mention the following seminal theorem, which was formerly called the conjecture of Taniyama, Shimura and Weil.

Theorem 2.14 (Wiles, Taylor, Breuil, Conrad, Diamond)**.** *Every elliptic curve, defined over $\mathbb{Q}$, is modular, i.e. it is associated with some 2-dimensional Hecke invariant piece $H_B^{(i)} \subseteq H^1(\overline{M_{0,N}}(\mathbb{C}), \mathbb{Q})$.*

More generally the Langlands program predicts a correspondence between modular forms associated with an arbitrary reductive group G and Galois representations (of type G). All concrete realizations of such correspondences, that have been constructed so far, occurred in the cohomology of Shimura varieties similar to the way described in this section. The matching of properties of the Galois representation on the one hand side, and the corresponding modular form on the other hand side, become much more involved, however, and beyond the scope of these lectures.

2.7 Outlook

2.15. In section 2.5 we explained that the natural generalizations of the modular curves are varieties of the form $\Gamma\backslash\mathbb{D}$ where $\mathbb{D}$ is a Hermitian symmetric domain and Γ is a congruence subgroup defined by some semi-simple $\mathbb{Q}$-algebraic group G such that $\mathbb{D}$ is a quotient of $G(\mathbb{R})$.

The datum $(G, \mathbb{D})$ is however slightly inconvenient to get a nice theory and also the description $\Gamma\backslash\mathbb{D}$ of the Shimura variety is slightly naive for arithmetic questions (e.g. to obtain a characterization of the model over number fields). Deligne [3, 4], in his synopsis of Shimura's work, gave an ingenious axiomatization of Shimura varieties which avoids these problems. The main goal of the lecture will be to understand all ideas behind his definition. We state the rough definition here for the purpose of illustration. All terms and the ideas behind the definition will be explained in detail during the lectures.

2.16. Deligne defines a **Shimura datum** as a pair consisting of a reductive algebraic group $\widetilde{G}$ (no longer semi-simple) defined over $\mathbb{Q}$ and a conjugacy class of morphisms

$$\widetilde{\mathbb{D}} \subset \mathrm{Hom}(\mathbb{S}, \widetilde{G}_{\mathbb{R}})$$

where $\mathbb{S}$ is the Deligne torus (the algebraic group over $\mathbb{R}$ with $\mathbb{S}(\mathbb{R}) = \mathbb{C}^*$) subject to the following conditions:

(SV1) $\mathrm{Ad} \circ h$ induces a representation of $\mathbb{S}$ on $\mathrm{Lie}(\widetilde{G}_{\mathbb{C}})$ with the weights $(-1,1),(0,0)$ and $(1,-1)$ for some (hence all) $h \in \mathbb{D}$;

(SV2) $\mathrm{int}(h(i))$ is a Cartan involution of $\widetilde{G}^{\mathrm{ad}}_{\mathbb{R}}$ for some (hence all) $h \in \mathbb{D}$;

(SV3) $\widetilde{G}^{\mathrm{ad}}$ has no factor H such that the projection of some (hence any) $h \in \mathbb{D}$ on $H_{\mathbb{R}}$ is trivial.

Under these conditions Deligne defines the **Shimura variety** associated with the datum $(\widetilde{G}, \widetilde{\mathbb{D}})$ and a compact open subgroup $K \subset \widetilde{G}(\mathbb{A}_f)$ ($\mathbb{A}_f :=$ finite adeles of $\mathbb{Q}$) as the set of double cosets

$$\mathrm{Sh}_K(\widetilde{G}, \widetilde{\mathbb{D}}) := \widetilde{G}(\mathbb{Q}) \backslash \widetilde{\mathbb{D}} \times \widetilde{G}(\mathbb{A}_f)/K.$$

Actually under the conditions (SV1–2), $\widetilde{\mathbb{D}}$ is a (bunch of copies of a) Hermitian symmetric domain and $\mathrm{Sh}_K(\widetilde{G}, \widetilde{\mathbb{D}})$ is a finite union of quotients $\Gamma \backslash \mathbb{D}$ considered before. Deligne saw, however, that this description (and the datum of the more general group $\widetilde{G}$) is necessary and sufficient to have a neat theory of *canonical models* over number fields. His version (and extension) of Shimura's theorems is:

Theorem 2.17. *There is a unique family of* **models** $M_K(\widetilde{G}, \widetilde{\mathbb{D}})$ *defined over the various reflex fields* $E(\widetilde{G}, \widetilde{\mathbb{D}})$ *(an explicit number field determined by the datum) which is* **canonical** *in the following sense:*

1. *All "Hecke operators", i.e. morphisms induced by multiplication with an adele from the right or change of the group K are defined over $E(\widetilde{G}, \widetilde{\mathbb{D}})$.*

2. *For each morphism of Shimura data $(\widetilde{G}_1, \widetilde{\mathbb{D}}_1) \to (\widetilde{G}_2, \widetilde{\mathbb{D}}_2)$ the corresponding morphism*

$$\mathrm{Sh}_{K \cap \widetilde{G}_1(\mathbb{A}_f)}(\widetilde{G}_1, \widetilde{\mathbb{D}}_1) \to \mathrm{Sh}_K(\widetilde{G}_2, \widetilde{\mathbb{D}}_2)$$

is defined over $E(\widetilde{G}_1, \widetilde{\mathbb{D}}_1)E(\widetilde{G}_2, \widetilde{\mathbb{D}}_2)$ w.r.t. the models $M_{K \cap \widetilde{G}_1(\mathbb{A}_f)}(\widetilde{G}_1, \widetilde{\mathbb{D}}_1)$ and $M_K(\widetilde{G}_2, \widetilde{\mathbb{D}}_2)$.

3. *If T is a $\mathbb{Q}$-torus and $h : \mathbb{S} \to T_{\mathbb{R}}$ a homomorphism, the associated Shimura variety*

$$T(\mathbb{Q}) \backslash \{h\} \times T(\mathbb{A}_f)/K$$

*is a finite set. Call $E = E(T,h)$. To give a model of the Shimura variety over E it suffices to describe the action of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ on those points. There is a certain group homomorphism of tori (***reflex norm***)*

$$N : \mathrm{Res}_{E|\mathbb{Q}}(\mathbb{G}_m) \to T$$

which describes this Galois action as the composition

$$\mathrm{Gal}(\overline{\mathbb{Q}}/E) \longrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}/E)^{\mathrm{ab}} \xrightarrow{\sim} \pi_0(E^* \backslash \mathbb{A}_E^*) \xrightarrow{N} \pi_0(T(\mathbb{Q}) \backslash T(\mathbb{A})) \xrightarrow{proj.} T(\mathbb{Q}) \backslash \{h\} \times T(\mathbb{A}_f)/K$$

+ multiplication. Here the isomorphism $\mathrm{Gal}(\overline{\mathbb{Q}}/E)^{\mathrm{ab}} \cong \pi_0(E^* \backslash \mathbb{A}_E^*)$ is the inverse of the reciprocity isomorphism of class field theory.

3 Abelian varieties

3.1 Motivation

Jacobians analytically
Different description of E by giving a complex structure on a $\mathbb{R}^2$ or a Hodge structure of weight 1.

3.2 Analytic theory

3.2.1 Complex tori

Theorem 3.1. *There is an equivalence of categories*

$$\left\{\begin{array}{c}\textit{compact connected}\\ \textit{complex Lie groups } X\\ \textit{(complex tori)}\end{array}\right\} \xleftrightarrow{\sim} \left\{\begin{array}{c}\underline{\textit{Objects:}}\ \Lambda\cong\mathbb{Z}^{2g}\ \textit{a lattice with}\\ \textit{complex structure } J \textit{ on } V:=\Lambda\otimes_{\mathbb{Z}}\mathbb{R}\\ \underline{\textit{Morphisms:}}\ \alpha:\Lambda\to\Lambda'\ \textit{s.t.}\\ \textit{such that } \alpha_{\mathbb{R}}:V\to V' \textit{ is complex linear.}\end{array}\right\}$$

$$\Lambda\backslash V \quad \leftarrow\!\shortmid \quad (\Lambda, J)$$

Note that a **complex structure** on a real vector space V may be equivalently given by:

1. An automorphism $J : V \to V$ such that $J^2 = -1$.
2. An action of $\mathbb{C}$ on V (i.e. a ring homomorphism $\mathbb{C} \to \mathrm{End}(V)$) coinciding on $\mathbb{R}$ with scalar multiplication.
3. A decomposition
$$V_{\mathbb{C}} = V^{-1,0} \oplus V^{0,-1}$$
such that $\overline{V^{-1,0}} = V^{0,-1}$ (a **Hodge structure of weight -1**)[4].

3.2. Let X be a compact complex Lie group, and denote $V := T_e(X)$ its Lie algebra. From the theory of Lie groups we get a unique holomorphic map

$$\exp : V \to X$$

with the properties

1. $\exp((\alpha+\beta)v) = \exp(\alpha v)\circ\exp(\beta v)$, for all $\alpha,\beta\in\mathbb{C}$ and $v \in X$,
2. $\exp(0) = e$,
3. $\mathrm{d}(\exp)_0 = \mathrm{id}_V$.

Lemma 3.3. *Let $\phi : X_1 \to X_2$ be a homomorphism of complex Lie groups with Lie algebras V_1, and V_2 respectively, then the following diagram is commutative*

$$\begin{tikzcd} V_1 \arrow[r, "(\mathrm{d}\phi)_e"] \arrow[d, "\exp"'] & V_2 \arrow[d, "\exp"] \\ X_1 \arrow[r, "\phi"] & X_2 \end{tikzcd}$$

Proof (Sketch) of theorem 3.1. We prove the essential surjectivity of the functor given in the statement of the theorem (thereby establishing that compact connected complex Lie groups are indeed commutative). Let X be compact complex connected Lie group of dimension g. Define

$$\begin{array}{rcl} C_x : X & \to & X,\\ y & \mapsto & xyx^{-1}. \end{array}$$

Note that $(\mathrm{d}C_x)_e : V \to V$ is an automorphism of the Lie algebra. We obtain a holomorphic map

$$\begin{array}{rcl} X & \to & \mathrm{Aut}(V)\\ x & \mapsto & (\mathrm{d}C_x)_e \end{array}$$

[4] We take weight -1 instead of $+1$ because we imagine V as $H_1(X,\mathbb{R})$, and want to adopt the convention the the weights of the Hodge structures on *cohomology* are positive.

By the maximum modulus principle this map has to be constant because X is compact. Therefore

$$(\mathrm{d}C_x)_e = \mathrm{id}_V \qquad \text{for all } x.$$

Applying the Lemma we get that

$$C_x(\exp(y)) = \exp((\mathrm{d}C_x)_e y) = \exp(y).$$

Therefore the image of exp lies in the center of X. Therefore exp is a group homomorphism (e.g. using the Baker-Campbell-Hausdorff formula). The image of exp is therefore open because $(\mathrm{d}\exp)_0$ has maximal rank, therefore also closed (cosets of the image are open, too). Therefore it has to be surjective, and the kernel is a lattice of rank $2g$ in V:

$$X \cong \Lambda\backslash V.$$

In particular $V \cong \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, hence, by choosing a basis of Λ, we arrive at the description given in the theorem. Fully faithfulness of the functor follows from Lemma 3.3. ☐

3.2.2 Cohomology

Theorem 3.4. *Let $X = \Lambda\backslash V$ be a complex torus. Then*

1. $\pi_1(X, e) = \Lambda$, *canonically.*
2. $H^1(X, \mathbb{Z}) = \mathrm{Hom}(\Lambda, \mathbb{Z})$
   $H^1(X, \mathbb{C}) = \mathrm{Hom}(\Lambda, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) \oplus \overline{\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})}$.
   Here $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is the space of complex linear maps $V \to \mathbb{C}$ and is also isomorphic to $H^0(X, \Omega^1)$ (space of global homomorphic 1-forms) and $\overline{\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})}$ is the space of complex anti-linear maps $V \to \mathbb{C}$ and is also isomorphic to $H^1(X, \mathcal{O})$.
3. $H^i(X, \mathbb{Z}) = \wedge^i H^1(X, \mathbb{Z})$ *as Hodge structures,*
   in particular

$$H^i(X, \Omega^j) \cong \wedge^i \mathrm{Hom}(V, \mathbb{C}) \otimes \wedge^j \overline{\mathrm{Hom}(V, \mathbb{C})}.$$

Proof. We have obviously $\pi_1(X, e) = \Lambda$, and therefore $H^1(X, \mathbb{Z}) = \mathrm{Hom}(\pi_1(X), \mathbb{Z}) = \mathrm{Hom}(\Lambda, \mathbb{Z})$.
Consider the map

$$\wedge^r(H^1(X, \mathbb{Z})) \to H^r(X, \mathbb{Z})$$

given by the cup-product. From the Künneth formula follows that if this is an isomorphism for all r for spaces X_1 and X_2 then also for all r and for $X = X_1 \times X_2$. Since our X is topologically isomorphic to $(S^1)^{2g}$, the statement 2. follows.
Now consider the sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O} \longrightarrow \Omega^1_c \longrightarrow 0$$

where Ω^1_c is the sheaf of *closed* holomorphic 1-forms. It induces a long exact sequence in cohomology:

$$H^0(X, \Omega^1) \cong H^0(X, \Omega^1_c) \longrightarrow H^1(X, \mathbb{C}) \longrightarrow H^1(X, \mathcal{O}) \longrightarrow H^1(X, \Omega^1_c)$$

A rather difficult analysis shows that the map $H^1(X, \mathcal{O}) \to H^1(X, \Omega^1_c)$ is zero, and therefore we get a decomposition (the Hodge decomposition):

$$H^1(X, \mathbb{C}) = H^0(X, \Omega^1) \oplus H^1(X, \mathcal{O}).$$

$H^0(X, \Omega^1)$ coincides with the space of *invariant* holomorphic 1-forms, which are automatically closed, and which are determined by its value at e and can therefore be seen as elements in $\mathrm{Hom}(V, \mathbb{C})$. ☐

3.2.3 Line bundles

Recall that not all complex analytic manifolds have algebraic models. In particular not all complex tori will have algebraic models. If they do have a model which is a projective variety then there must be necessarily line bundles with sufficiently many sections to construct an embedding into a projective space. In this section we classify line bundles on a complex torus.
For a quotient like $X = \Lambda\backslash V$ there is a strong relation between the cohomology on V and the group cohomology of the group Λ.
We begin by briefly reviewing group cohomology. Let M be a Λ-module.
Define the **complex of Λ-cocycles with values in** M to be the complex

$$0 \longrightarrow C^0(\Lambda, M) \xrightarrow{d} C^1(\Lambda, M) \xrightarrow{d} C^2(\Lambda, M) \xrightarrow{d} \cdots \tag{5}$$

where

$$C^p(\Lambda, M) := \{p : \Lambda^p \to M\}$$

and

$$df(\sigma_0, \ldots, \sigma_p) = \sigma_0 \cdot f(\sigma_1, \ldots, \sigma_p) + \sum_{i=0}^{p}(-1)^i f(\sigma_0, \ldots, \sigma_i\sigma_{i+1}, \ldots, \sigma_p) + (-1)^{p+1} f(\sigma_0, \ldots, \sigma_{p-1}).$$

Definition 3.5. *The cohomology groups $H^p(\Lambda, M) = Z^p(\Lambda, M)/\mathrm{d}(C^{p-1}(\Lambda, M))$ of the complex (5) are called the* **group cohomology** *of the module M.*

They coincide with the right derived functors of the functor

$$M \mapsto M^\Lambda = \{m \in M \mid \lambda m = m \ \forall \lambda \in \Lambda\}.$$

For a bilinear pairing:

$$\langle , \rangle : M \times N \to P$$

we can define a cup-product

$$\cup : H^p(\Lambda, M) \times H^q(\Lambda, N) \to H^{p+q}(\Lambda, P)$$

given explicitly by

$$(f \cup g)(\sigma_1, \ldots, \sigma_{p+q}) = \langle f(\sigma_1, \ldots, \sigma_p), g(\sigma_{p+1}, \ldots, \sigma_{p+q})\rangle.$$

3.6. Let V be a manifold on which a group Λ acts freely and discontinously (i.e. the action is proper for the discrete topology on Λ). Let

$$\pi : V \to \Lambda\backslash V$$

be the projection onto the quotient.

Proposition 3.7. *For every sheaf of abelian groups $\mathcal{F}$ on X and every p there exists a natural map*

$$\phi_p : H^p(\Lambda, \Gamma(V, \pi^*\mathcal{F})) \to H^p(X, \mathcal{F})$$

with the properties:

1. *compatibility with long exact sequences*
2. *compatibility with cup products*
3. *If $H^i(V, \pi^*\mathcal{F}) = 0$ for all $i \geq 1$ then ϕ_p is an isomorphism.*

Proof idea: There exists a covering $\{V_i\}$ of X such that

1. $\pi^{-1}(V_i) = \bigcup_{\sigma\in\Lambda} \sigma(U_i) \qquad U_i \subset V$ open, s.t. $\pi : U_i \xrightarrow{\sim} V_i$,
2. $\forall i, j$ $\exists$ at most one σ s.t. $U_j \cap \sigma(U_i) \neq \emptyset$.

Let $\sigma_{i,j}$ denote the σ of property 2., if it exists.
Define a map

$$\begin{aligned} \phi_p : C^p(\Lambda, \Gamma(\pi^*\mathcal{F})) &\to C^p(\{V_i\}, \mathcal{F}), \\ (\phi_p f)_{i_0,\dots,i_p} &= f(\sigma_{i_0,i_1}, \dots, \sigma_{i_{p-1},i_p}) \quad \substack{\text{restricted to} \\ U_{i_0}\cap\cdots\cap U_{i_p}} \end{aligned}$$

One checks that this defines a morphism of complexes and can elementarily check the properties 1.–3. In more fancy terms we are looking at the Leray spectral sequence for the following composition of morphisms of (Artin) stacks:

$$\begin{tikzcd} {[V/\Lambda]} \arrow[rr, "\sim"] \arrow[d, "p_V"'] & & X \arrow[d, "p_X"] \\ B\Lambda := [\cdot/\Lambda] \arrow[rr, "\pi"'] & & \cdot \end{tikzcd}$$

where we have

$$\begin{aligned} R^i p_{V,*} &= \text{sheaf cohomology on } V\text{, with its } \Lambda\text{-action}, \\ R^i p_{X,*} &= \text{sheaf cohomology on } X, \\ R^i \pi_* &= \text{group cohomology of } \Lambda. \end{aligned}$$

□

Note that

$$H^1(V, \mathcal{O}^*) \quad \text{and} \quad H^1(X, \mathcal{O}^*)$$

classify line bundles on V, and on X, respectively. It is a classical fact that $H^1(V, \mathcal{O}^*) = \{1\}$. A refinement of Proposition 3.7 shows that therefore that the top horizontal homomorphism in the following diagram is an isomorphism:

$$\begin{tikzcd} H^1(\Lambda, \mathcal{O}^*(V)) \arrow[r, "\sim"] \arrow[d, "2"', "\sim"] & H^1(X, \mathcal{O}^*) \arrow[d, "3"', "\sim"] \\ \left\{\substack{\Lambda-\text{equivariant} \\ \text{line-bundles on } V}\right\}/\sim \arrow[r, "\sim", "1"'] & \{\text{line bundles on } X\}/\sim \end{tikzcd} \tag{6}$$

We can also describe the other maps in the diagram explicitly:

1 This map associates with a Λ-equivariant line bundle $\widetilde{\mathcal{L}} \to V$ the line bundle on X given by

$$\Lambda \backslash \widetilde{\mathcal{L}} \to \Lambda \backslash V = X.$$

2 Choose a trivialization $\widetilde{\mathcal{L}} \cong \mathbb{C} \times V$ as line bundle (not equivariantly). It yields a Λ-action on $\mathbb{C} \times V$. Any such Λ-action may be described by

$$\sigma(z, v) = (f_\sigma(v) \cdot z, \sigma(v))$$

where $f_\sigma \in H^0(V, \mathcal{O}^*(V))$ satsifying

$$f_\tau(\sigma(v)) f_\sigma(v) = f_{\tau\sigma}(v)$$

or equivalently

$$f \in C^1(\Lambda, \mathcal{O}^*(V)) \qquad \mathrm{d}f = 0.$$

Boundaries in $C^1(\Lambda, \mathcal{O}^*(V))$ give isomorphic Λ-equivariant bundles, hence we established an isomorphism 2.

3 Trivializing

$$\alpha_i : \mathcal{L}|_{U_i} \cong \mathbb{C} \times U_i$$

gives a cocycle

$$\{\alpha_i \alpha_j^{-1}\}_{i,j} \in H^1(\{U_i\}, \mathcal{O}^*) \cong H^1(X, \mathcal{O}^*).$$

Exercise 3.8. *Show that the diagram (6) is commutative.*

Let now V be a complex vector space and $L \subset V$ a lattice of full rank. Consider the exponential sequence of sheaves of Abelian groups on $X = \Lambda \backslash V$:

$$0 \longrightarrow \mathbb{Z}(1) \longrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 1$$

Here we denote $\mathbb{Z}(1) = 2\pi i \mathbb{Z} \subset \mathbb{C}$. We have

$$H^i(V, \mathbb{Z}(1)) = 0 \qquad H^i(V, \mathcal{O}) = 0 \quad \forall\, i \geq 1$$

hence by Proposition 3.7 we can identifiy the two associated long exact sequences

$$\begin{tikzcd}
 & & & & Z^2(\Lambda,\mathbb{Z}(1)) \arrow[d] & \\
0 \arrow[r] & H^1(\Lambda,\mathbb{Z}(1)) \arrow[r] \arrow[d, "\sim"] & H^1(\Lambda,\mathcal{O}(V)) \arrow[r] \arrow[d, "\sim"] & H^1(\Lambda,\mathcal{O}^*(V)) \arrow[r, "\delta"] \arrow[d, "\sim"] & H^2(\Lambda,\mathbb{Z}(1)) \arrow[r] \arrow[d, "\sim"] & H^2(\Lambda,\mathcal{O}(V)) \arrow[d, "\sim"] \\
0 \arrow[r] & H^1(X,\mathbb{Z}(1)) \arrow[r] \arrow[d, "\sim"] & H^1(X,\mathcal{O}) \arrow[r] \arrow[d, "\sim"] & H^1(X,\mathcal{O}^*) \arrow[r, "\delta"] & H^2(X,\mathbb{Z}(1)) \arrow[r] \arrow[d, "\sim"] & H^2(X,\mathcal{O}) \arrow[d, "\sim"] \\
 & \mathrm{Hom}(\Lambda,\mathbb{Z}(1)) \arrow[r] & \overline{\mathrm{Hom}(V,\mathbb{C})} & & \underset{\text{Hodge structure of wt. 2}}{\wedge^2(\mathrm{Hom}(\Lambda,\mathbb{Z}))(1)} \arrow[r] & \underset{(0,2)-\text{component}}{\wedge^2\overline{\mathrm{Hom}(V,\mathbb{C})}}
\end{tikzcd}$$

We have to understand the right hand side of this diagram in more detail:
Let $\mathcal{L}$ be a line bundle given by $f \in Z^1(\Lambda, \mathcal{O}^*(V))$. Then there are $g_\sigma \in \mathcal{O}(V)$ such that

$$f_\sigma = \exp g_\sigma$$

and by definition

$$\delta(f)_{\sigma,\tau} = g_{\sigma+\tau}(z) - g_\tau(z+\sigma) - g_\sigma(z).$$

Furthermore, one can show that the long vertical composition is given as follows:

$$\begin{array}{rcl} Z^2(\Lambda, \mathbb{Z}(1)) & \to & \mathrm{Hom}(\wedge^2 \Lambda, \mathbb{Z}) \\ f & \mapsto & \{\sigma, \tau \mapsto f_{\sigma,\tau} - f_{\tau,\sigma}\} \end{array}$$

We know from the diagram that is is surjective with kernel $\mathrm{d}(Z^1(\Lambda, \mathbb{Z}(1)))$. It is an exercise to verify this elementarily.
Furthermore

$$\wedge^2(\mathrm{Hom}(\Lambda, \mathbb{C})) = \wedge^2(\mathrm{Hom}(V, \mathbb{C})) \oplus \mathrm{Hom}(V, \mathbb{C}) \otimes \overline{\mathrm{Hom}(V, \mathbb{C})} \oplus \wedge^2(\overline{\mathrm{Hom}(V, \mathbb{C})})$$

is a Hodge structure of weight 2. For elements in $\wedge^2(\mathrm{Hom}(\Lambda, \mathbb{Z}(1)))$ whose $(0,2)$-component vanishes, the $(2,0)$-component automatically also vanishes because they are real. Hence the image of δ lies in the subspace

$$NS(X) := \wedge^2(\mathrm{Hom}(\Lambda, \mathbb{Z}(1))) \cap \mathrm{Hom}(V, \mathbb{C}) \otimes \overline{\mathrm{Hom}(V, \mathbb{C})}.$$

Lemma 3.9. *If we consider elements in $\wedge^2(\mathrm{Hom}(\Lambda,\mathbb{Z}))(1) = \mathrm{Hom}(\wedge^2\Lambda,\mathbb{Z}(1))$ as alternating forms on Λ, then the condition $E \in \mathrm{NS}(X)$ can be stated as*

$$E(iv, iw) = E(v, w)$$

where E has been extended linearly to $V = \Lambda \otimes \mathbb{R}$ and multiplication by i refers to the given complex structure.

Proof. The Hodge decomposition is compatible with considering elements as alternating functions on $\Lambda \otimes \mathbb{R}$, i.e. an element $f \wedge g \in \wedge^2 \mathrm{Hom}(V,\mathbb{C})$ (resp. $\in \mathrm{Hom}(V,\mathbb{C}) \otimes \overline{\mathrm{Hom}(V,\mathbb{C})}$, resp. $\in \wedge^2\overline{\mathrm{Hom}(V,\mathbb{C})}$) is send to the function on $\wedge^2\Lambda \otimes \mathbb{R}$ given by:

$$v \wedge w \mapsto f(v)g(w) - f(w)g(v).$$

The tensors in the $(0,2)$- and $(2,0)$-component therefore satisfy the equation

$$E(iv, iw) = -E(v, w)$$

and the tensors in the $(1,1)$-component satisfy

$$E(iv, iw) = E(v, w)$$

□

Hence δ is a surjective map:

$$\begin{array}{ccc} H^1(\Lambda, \mathcal{O}^*(V)) & \xrightarrow{\delta} & \mathrm{NS}(X) = \left\{E \in \mathrm{Hom}(\wedge^2\Lambda, \mathbb{Z}(1)) \mid E(i\sigma, i\tau) = E(\sigma,\tau)\right\} \\ \{f_\sigma\} & \mapsto & \{\sigma, \tau \mapsto g_\sigma(z+\tau) - g_\tau(z+\sigma) + g_\tau(z) - g_\sigma(z)\} \end{array}$$

The condition $E(i\sigma, i\tau) = E(\sigma,\tau)$ is equivalent to the property that $\frac{1}{2\pi i}E$ is the imaginary part of a Hermitian bilinear form, which may be reconstructed as

$$H(v, w) := \frac{1}{2\pi i}(E(v, iw) + iE(v, w)).$$

The quotient

$$\mathrm{Hom}(\Lambda, \mathbb{Z}(1)) \backslash \overline{\mathrm{Hom}(V, \mathbb{C})}$$

is itself a complex torus which we call $X^\vee$, the dual of X. An element $\alpha \in \overline{\mathrm{Hom}(V,\mathbb{C})}$ is send to the cocycle

$$\tau \mapsto \exp(\alpha(\tau))$$

in $H^1(\Lambda, \mathcal{O}^*(V))$ (it has values even in constant functions). Because α is complex anti-linear, this is in general *not* a trivial cocycle.

The remaining task is to explicitly represent cocycles which are not in the image of this map. After constant functions, the next try is to insert functions which are exponentials of linear functions:

$$f_\sigma := \exp(l_\sigma \cdot z + a_\sigma).$$

The cocycle condition

$$f_{\sigma+\tau}(z) f_\sigma(z+\tau)^{-1} f_\tau(z)^{-1} \equiv 1$$

hence boils down to

$$l_{\sigma+\tau}z + a_{\sigma+\tau} - l_\sigma(z+\tau) - a_\sigma - l_\tau(z) - a_\tau \in \mathbb{Z}(1)$$

or

$$l_{\sigma+\tau} - l_\sigma - l_\tau = 0 \quad \text{and} \quad a_{\sigma+\tau} - a_\sigma - a_\tau \equiv l_\sigma \tau \mod \mathbb{Z}(1).$$

We get

$$\begin{aligned} (\delta f)_{\sigma,\tau} &= l_\sigma(z+\tau) - l_\tau(z+\sigma) + l_\tau z - l_\sigma z \\ &= l_\sigma\tau - l_\tau\sigma. \end{aligned}$$

Hence, to get a preimage for E, we are looking for a linear function

$$l : \Lambda \to \mathbb{C}$$

such that

$$E(\sigma, \tau) = l(\sigma)\tau - l(\tau)\sigma.$$

Such a function is given by the linear function determined by:

$$l(\sigma)z = \pi H(z, \sigma)$$

because

$$l(\sigma)\tau - l(\tau)\sigma = \pi(H(\tau, \sigma) - H(\sigma, \tau)) = E(\sigma, \tau).$$

To see that these exponentials of linear functions are sufficient, we have to verify that the equation

$$a_{\sigma+\tau} - a_\sigma - a_\tau \equiv \pi H(\tau, \sigma) \mod \mathbb{Z}(1)$$

can always be solved. Trick: Define $\widetilde{\alpha}_\sigma := a_\sigma - 2\pi H(\sigma, \sigma)$. We have then

$$\widetilde{a}_{\sigma+\tau} - \widetilde{a}_\sigma - \widetilde{a}_\tau + 2\pi H(\sigma+\tau, \sigma+\tau) - 2\pi H(\sigma, \sigma) - 2\pi H(\tau, \tau) \equiv 2\pi H(\tau, \sigma) \mod \mathbb{Z}(1)$$
$$\widetilde{a}_{\sigma+\tau} - \widetilde{a}_\sigma - \widetilde{a}_\tau + 2\pi(H(\sigma, \tau) - H(\tau, \sigma)) \equiv 0 \mod \mathbb{Z}(1)$$
$$\widetilde{a}_{\sigma+\tau} - \widetilde{a}_\sigma - \widetilde{a}_\tau + \frac{1}{2}E(\sigma, \tau) \equiv 0 \mod \mathbb{Z}(1)$$

The term $\frac{1}{2}E(\sigma, \tau)$ vanishes in this congruence if E is even. If E is not even the equation can be solved, too. We summarize everything by

Proposition 3.10 (Appel-Humbert)**.** *We have a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & X^\vee \arrow[r] \arrow[d, "\sim"] & H^1(X, \mathcal{O}^*) \arrow[r] \arrow[d, "\sim"] & \mathrm{NS}(X) \arrow[r] \arrow[d, "\sim"] & 0 \\
0 \arrow[r] & \left\{\widetilde{\alpha} \in \mathrm{Hom}(\Lambda, \mathbb{Z}(1)) \backslash \overline{\mathrm{Hom}(V, \mathbb{C})}\right\} \arrow[r] & \left\{ {(E,\widetilde{\alpha}) \atop \frac{1}{2}E(\sigma,\tau) + \widetilde{\alpha}_{\sigma+\tau} - \widetilde{\alpha}_\sigma - \widetilde{\alpha}_\tau \in \mathbb{Z}(1)} \right\}_{/\sim} \arrow[r] & \left\{ {E \in \mathrm{Hom}(\wedge^2\Lambda, \mathbb{Z}(1)) \atop E(i\sigma, i\tau) = E(\sigma, \tau)} \right\} \arrow[r] & 0
\end{tikzcd}$$

whose rows are exact sequences.

Here the maps in the second line are given by

$$\widetilde{\alpha} \mapsto (0, \widetilde{\alpha}) \quad \text{and} \quad (E, \widetilde{\alpha}) \mapsto E,$$

respectively.
$\widetilde{\alpha}$ in the middle set is a function $\Lambda \to \mathbb{C}$ that satisfies the relation given. There is an equivalence relation on the $\widetilde{\alpha}$ which has to be divided out; we will not make this explicit.

3.11. We denote the line bundle determined by a pair $(E, \widetilde{\alpha})$ by $\mathcal{L}(E, \widetilde{\alpha})$. By the calculations above, it is given by the cocycle

$$f_\sigma := \exp\left(\pi H(z + \frac{1}{2}\sigma, \sigma) + \widetilde{\alpha}(\sigma)\right).$$

Observer also that, up to inverting 2, the map δ has a canonical section $\frac{1}{2}s$, where

$$s(E) = \mathcal{L}(2E, 0).$$

From the diagram (6) we may infer that we have

$$H^0(X, \mathcal{L}(E, \widetilde{\alpha})) = \left\{ \begin{array}{c} f : V \to \mathbb{C} \text{ holomorphic} \\ \text{s.t. } f(z+\sigma) = f_\sigma f(z) \end{array} \right\}$$

for the function f_σ above. These function are called **theta functions**.

More difficult is the following theorem of Riemann whose proof we won't discuss.

Theorem 3.12. *If H is* positive definite *then*

1. $\dim H^0(X, \mathcal{L}(E, \widetilde{\alpha})) = \sqrt{\det(E)}$,
2. $\mathcal{L}(E, \widetilde{\alpha})^{\otimes 3}$ *defines an embedding $X \hookrightarrow \mathbb{P}^N(\mathbb{C})$ for some N.*

By Chow's theorem we have therefore:

Corollary 3.13. *X is a complex projective algebraic variety if and only if there exists an $E \in \mathrm{NS}(X)$ such that that the corresponding Hermitian form H is positive definite.*

The "only if" part follows because one can show that for any projective embedding $\varphi : X \hookrightarrow \mathbb{P}^N$ the associated ample line bundle $\varphi^{-1}\mathcal{O}(1)$ has a non-zero class $E \in \mathrm{NS}(X)$ such that that the corresponding Hermitian form H is positive definite.
If V has complex dimension 1, then there are 2 generators $E^{\pm}$ of $\mathrm{Hom}(\wedge^2\Lambda, \mathbb{Z}(1))$ which are automatically in $\mathrm{NS}(X)$, and for one of them, E^+ say, the associated Hermitian form will be positive definite. This reproves that all complex tori of dimension 1 are algebraic. The functions in $H^0(X, \mathcal{L}(E^+, \widetilde{\alpha}))$ are the classical Riemann theta functions.

3.2.4 Moduli

Riemann's theorem 3.12 motivates the following definition:

Definition 3.14. *Let X be a complex torus. A symplectic form $E \in \mathrm{NS}(X)$ s.t. the corresponding Hermitian form H is (positive of negative) definite is called a* **polarization** *of X.*

As for elliptic curves, we are interested in a moduli space for Abelian varieties (= polarizable complex tori). It turns out that the naive moduli problem is not well behaved, for example, due to the fact that even with level structures of arbitrarily high degree, there still exist automorphisms. The situation is much better when we try to classify Abelian varieties *together with one of their polarizations*, which in view of Riemann's theorem is almost the same as considering X together with (a $\mathrm{PGL}(N)$-orbit) of a fixed embedding into some $\mathbb{P}^N$. (Strictly speaking, this is only true, if the polarizations are sufficiently divisible, e.g. by 3). It is also convenient to group the moduli problems according to invariants of symplectic forms on lattices. Recall that for a non-degenerate symplectic form E on a lattice $\Lambda \cong \mathbb{Z}^{2g}$ there exists a basis of Λ such that the form has the matrix

$$\phi_d := \begin{pmatrix} & & & d_1 & & \\ & & & & \ddots & \\ & & & & & d_g \\ -d_1 & & & & & \\ & \ddots & & & & \\ & & -d_g & & & \end{pmatrix} \tag{7}$$

where $d_i \in \mathbb{Z}_{>0}$ and $d_1|d_2|\cdots|d_g$. The d_i's are uniquely determined by E. We will call $d = (d_1, \ldots, d_g)$ the **type** of E.

3.15. For a type $d = (d_1, \ldots, d_g)$ consider the set

$$\mathcal{A}_{g,d} := \left\{ \begin{array}{c} X \text{ complex torus of dim } g \\ E \in \mathrm{NS}(X) \text{ a polarization of type } d \end{array} \right\}_{/ \text{ Iso.}}$$

Since Λ is canonically isomorphic to $H_1(X, \mathbb{Z})$ we may choose an isomorphism

$$\beta : H_1(X, \mathbb{Z}) \to \mathbb{Z}^{2g}$$

such that β transports the symplectic form E on the left hand side to the form given by $\pm 2\pi i$ times the matrix (7) on the right hand side, i.e. β is an (integral) symplectic similitude. Any two such isomorphisms differ by an element of the symplectic similitude group $\mathrm{GSp}(2g, \mathbb{Z})$ (acting by pre-composition).

It follows that $\mathcal{A}_{g,d}$ is the quotient modulo $\mathrm{GSp}(2g,\mathbb{Z})$ of the following set:

$$\mathbb{H}_g^{\pm} := \left\{ \begin{array}{c} X \text{ complex torus of dim } g \\ E \in \mathrm{NS}(X) \text{ a polarization of type } d \\ \beta : H_1(X,\mathbb{Z}) \to \mathbb{Z}^{2g} \end{array} \right\}_{/ \text{ Iso.}}$$

We will see that $\mathbb{H}_g^{\pm}$ is nothing but Siegel's upper (and lower) half space mentioned in the introduction. For $g=1$ we get back the upper and lower half planes $\mathbb{H}^{\pm} \subset \mathbb{C}$.
Using Theorem 3.1 we can translate this into

$$\mathbb{H}_g^{\pm} = \left\{ \begin{array}{c} \text{complex structure on } V = \mathbb{R}^{2g} \\ \text{s.t. } \mathbb{Z}^{2g}\backslash V \text{ is polarized via } \pm 2\pi i \phi_d \end{array} \right\}$$

where ϕ_d was defined in (7).

Proof. A complex torus is (up to isomorphism) given by a lattice Λ together with a Hodge structure on $V = \Lambda_{\mathbb{R}}$. The polarization is an element of $\wedge^2\Lambda(1)$. The given β translates this into the standard lattice such that the polarization becomes ϕ_d. By transport of structure along β, we get a complex structure on $\mathbb{R}^{2g}$. Conversely given a complex structure on $V = \mathbb{R}^{2g}$ such that ϕ_d is a polarization for $\mathbb{Z}^{2g}\backslash V$ there exists the canonical $\beta : H_1(X,\mathbb{Z}) \to \mathbb{Z}^{2g}$ given by the identity. $\square$

We start by investigating the condition $\phi_d(iv, iw) = \phi_d(v,w)$ from the viewpoint of the complex structure. For the purpose of later generalizations, we make a little digression on more general Hodge structures. We already saw that it is equivalent to the condition $\phi_d \in (V^* \otimes V^*)^{1,1}$ w.r.t. the natural Hodge structure induces by the complex structure on $V = \mathbb{R}^{2g}$.

3.16. Recall that a complex structure on V is the same as a Hodge structure[5] of weight -1, i.e. a decomposition

$$V_{\mathbb{C}} = V^{-1,0} \oplus V^{0,-1}$$

such that $\overline{V^{-1,0}} = V^{0,-1}$. Such a decomposition is also the same as a certain representation of the algebraic group (torus) $\mathbb{S} = \mathrm{Res}_{\mathbb{R}}^{\mathbb{C}}(\mathbb{G}_m)$. This latter group may be described as the algebraic group $\mathbb{S} \subset \mathrm{GL}_{2,\mathbb{R}}$ given by matrices of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

$\mathbb{S}_{\mathbb{C}}$ is isomorphic to $\mathbb{G}_{m,\mathbb{C}}^2$ by virtue of[6]

$$\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \begin{pmatrix} a+bi & 0 \\ 0 & a-bi \end{pmatrix}.$$

An algebraic representation of $\mathbb{G}_{m,\mathbb{C}}^2$ on $V_{\mathbb{C}}$ is the same as giving a bigrading

$$V_{\mathbb{C}} = \oplus_{p,q} V^{p,q} \tag{8}$$

by virtue of $V^{p,q} = \{v \in V_{\mathbb{C}} \mid (z_1,z_2)v = z_1^p z_2^q v\}$. If we identify $\mathbb{S}(\mathbb{R}) = \mathbb{C}^*$ by means of $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mapsto z := a+bi$ then we have

$$V^{p,q} = \{v \in V_{\mathbb{C}} \mid \begin{pmatrix} a & b \\ -b & a \end{pmatrix} v = z^p \overline{z}^q v\}.$$

From this, it follows that any algebraic representation of $\mathbb{S}$ on a real vector space V is equivalent to a decomposition (8) of $V_{\mathbb{C}}$ with the additional property

$$V^{p,q} = \overline{V^{q,p}}.$$

[5] More precisely: a Hodge structure of type $(-1,0),(0,-1)$.
[6] More generally, all tori over an algebraically closed field are isomorphic to a power of $\mathbb{G}_m$.

In other words, we have an equivalence of categories

$$\{\text{real Hodge structures}\} \cong \{\text{f.d. algebraic representations of } \mathbb{S}\}.$$

We will say that an object on either side is of type $(p_1, q_1), \ldots, (p_n, q_n)$ if $V^{p_1,q_1}, \ldots, V^{p_n,q_n}$ are the only non-zero spaces in the decomposition (8). Thus, for instance, a complex structure is the same as a representation of $\mathbb{S}$ of type $(-1, 0), (0, -1)$.

Lemma 3.17. *Let (V, ϕ), where $V = \mathbb{R}^{2g}$ and $\phi = \phi_d$ be our standard real symplectic vector space of type d. For a complex structure on V given by a representation $h : \mathbb{S} \to \mathrm{GL}(V)$, or equivalently by the endomorphism $J = h(-i)$, or equivalently by a decomposition $V_{\mathbb{C}} = V^{-1,0} \oplus V^{0,-1}$, the following are equivalent:*

1. *$\phi \in (V^* \otimes V^*)^{1,1}$ (or $\phi(Jv, Jw) = \phi(v, w)$ on $\mathbb{R}^{2g}$, where J is the complex structure),*
2. *$V^{-1,0}$ is isotropic (via the complex linear extension of ϕ to $\mathbb{C}^{2g}$).*
3. *h factors through $\mathrm{GSp}(\phi)$.*

If there is a basis of $V^{-1,0}$ of the form

$$\begin{pmatrix} 1_g \\ D^{-1}M \end{pmatrix}$$

where D is the diagonal matrix with entries $d_1, \ldots, d_g$, and where $M \in \mathrm{M}_{g\times g}(\mathbb{C})$ is some matrix, then the condition is furthermore equivalent to:

4. *${}^tM = M$.*

Proof. 1. $\Rightarrow$ 2. We have $\phi(Jv, Jw) = \phi(v, w)$, hence this holds also for the complex linear extension. Let $v, w \in V^{-1,0}$. We have

$$\phi(v, w) = \phi(Jv, Jw) = \phi(iv, iw) = i^2\phi(v, w).$$

Hence this quantity is 0.
2. $\Rightarrow$ 3. We have $V_{\mathbb{C}} = V^{-1,0} \oplus V^{0,-1}$ a decomposition into 2 isotropic subspaces. Hence the symplectic form is of the form

$$\phi((a, b), (c, d)) = \phi(a, d) + \phi(b, c)$$

We have

$$\phi((z_1 a, z_2 b), (z_1 c, z_2 d)) = z_1^{-1} z_2^{-1}(\phi(a, d) + \phi(b, c))$$

This shows that the corresponding representation of $\mathbb{S}$ leaves the symplectic form invariant up to scalar.
3. $\Rightarrow$ 1. Consider the induced representation of $\mathbb{S}$ on $V^* \otimes V^*$ which is of type $(0, 2), (1, 1), (2, 0)$. Because h factors through GSp, the form ϕ_d is multiplied by a scalar by the action of $\mathbb{S}$, hence must lie in the $(1, 1)$-component.
4. $\Leftrightarrow$ 2. Assume that $V^{-1,0}$ is given by the basis $\begin{pmatrix} 1_g \\ D^{-1}M \end{pmatrix}$. Its isotropy is then equivalent to

$$\begin{pmatrix} 1_g & D^{-1\,t}M \end{pmatrix} \begin{pmatrix} & D \\ -D & \end{pmatrix} \begin{pmatrix} 1_g \\ D^{-1}M \end{pmatrix} = 0$$

hence to

$${}^tM = M.$$

□

If the conditions of the Lemma hold, we say that **the complex structure is compatible with the symplectic structure**.

Lemma 3.18. *Let (V, ϕ), where $V = \mathbb{R}^{2g}$ and $\phi = \phi_d$ be our standard real symplectic vector space of type d. Assume we have a complex structure (given by $V_{\mathbb{C}} = V^{-1,0} \oplus V^{0,-1}$, or by some h) that is compatible with the symplectic structure, i.e. such that the equivalent conditions of the previous lemma hold. Then the following are equivalent*

1. *$\pm 2\pi i\phi_d$ is a polarization, i.e. the corresponding Hermitian form H is positive definite, or equivalently: The (now symmetric) form*

$$v, w \mapsto \phi(v, h(i)w)$$

is positive or negative definite.

2. *There is a basis of $V^{-1,0}$ of the form*

$$\begin{pmatrix} 1_g \\ D^{-1}M \end{pmatrix}$$

such that ${}^tM = M$ and such that the imaginary part of M is (positive or negative) definite.

3. *The stabilizer of the complex structure under translation by $\mathrm{Sp}(\mathbb{R}^{2g}, \phi_d)$ is maximal compact in the latter group.*

Proof. 1. $\Leftrightarrow$ 2. We first choose a basis for $V^{-1,0}$ of the form $\begin{pmatrix} M_1 \\ D^{-1}M_2 \end{pmatrix}$ where M_1, M_2 are in $\mathrm{M}_{g\times g}(\mathbb{C})$. Isotropy implies that

$$\begin{pmatrix} {}^tM_1 & D^{-1t}M_2 \end{pmatrix} \begin{pmatrix} & D \\ -D & \end{pmatrix} \begin{pmatrix} M_1 \\ D^{-1}M_2 \end{pmatrix} = 0.$$

Hence

$${}^tM_2 M_1 = {}^tM_1 M_2.$$

The condition from 1. implies for any $z \in V^{-1,0}$,

$$\phi_d(z + \overline{z}, h(i)(z + \overline{z})) = 2i\phi_d(z, \overline{z}) > 0.$$

(or always < 0). Hence the matrix

$$2i \begin{pmatrix} {}^tM_1 & D^{-1t}M_2 \end{pmatrix} \begin{pmatrix} & D \\ -D & \end{pmatrix} \begin{pmatrix} \overline{M_1} \\ D^{-1}\overline{M_2} \end{pmatrix} = 2i({}^tM_2\overline{M_1} - {}^tM_1\overline{M_2})$$

is (positive or negative) definite. Hence M_1 is invertible, because for $z \in \mathbb{C}^g$, $M_1 z = 0$ implies ${}^tz^tM_1 = \overline{M_1}\overline{z} = 0$ and therefore

$$2i({}^tz({}^tM_2\overline{M_1} - {}^tM_1\overline{M_2})\overline{z}) = 0$$

and therefore $z = 0$. Therefore multiplying with M_1^{-1} from the right, we see that $V^{-1,0}$ has a basis of the form:

$$\begin{pmatrix} 1_g \\ D^{-1}M \end{pmatrix} = 0$$

and the equation reduces to $2i(\overline{M_1} - M_1)$ positive or negative definite.
The converse follows by the same calculation.
1. $\Leftrightarrow$ 3. The stabilizer of the complex structure in $\mathrm{Sp}_{2g}(\mathbb{R})$ is precisely the unitary group of the associated Hermitian form. It is maximal compact in $\mathrm{Sp}_{2g}(\mathbb{R})$ if the Hermitian form is positive or negative definite. For the converse, not that the unitary group of an indefinite Hermitian form H cannot be compact: Each isotropic vector v defines unipotent elements in the real algebraic group $U_g(H)$. $\square$

Corollary 3.19.

$$\begin{aligned}
\mathbb{H}_g^{\pm} &\cong \{M \in \mathrm{M}_{g\times g}(\mathbb{C}) \mid {}^tM = M,\ \Im(M) \textit{ definite}\} && (9)\\
&\cong \mathrm{GSp}(\mathbb{R})/\operatorname{Stab}(h) \qquad \textit{where } \operatorname{Stab}(h) \textit{ is maximal compact modulo the center} && (10)\\
\mathcal{A}_{g,d} &\cong \mathrm{GSp}(\phi_d, \mathbb{Z})\backslash\mathbb{H}_g^{\pm} && (11)\\
\mathbb{H}_g^{\pm} &\hookrightarrow \{V^{-1,0} \subset \mathbb{C}^{2g} \textit{ isotropic subspace}\} \qquad \textit{open embedding into a complex projective variety} && (12)
\end{aligned}$$

In the second line h is given by the choice of any reference complex structure which satisfies the conditions of the Lemmas 3.17–3.18.

Proof. The identification (9) exists because we have seen that complex structures on $\mathbb{R}^{2g}$ which satisfy the conditions of Lemmas 3.17–3.18 are given by matrices M satisfying the conditions ${}^tM = M$ and $\Im(M)$ definite.
The identification (10) exists if $\mathrm{GSp}(\mathbb{R})$ (or even $\mathrm{Sp}(\mathbb{R})$) acts transitively on the set of complex structures, which satisfy the conditions of Lemmas 3.17–3.18. This follows from the fact that the stabilizer is maximal compact and any two maximal compact subgroups are conjugated in $\mathrm{Sp}(\mathbb{R})$. Alternatively one can show transitivity by an elementary calculation using the explicit description as Siegel upper half space (9).
We have seen the identification (11) before.
The embedding (12) is given by associating a complex structure, which satisfies the conditions of Lemmas 3.17–3.18, to its $V^{-1,0}$, which is an isotropic subspace. This is an embedding because the subspace $V^{-1,0}$ determines the complex structure and the image is open because the conditions that, (a) there exists a basis of the form $\begin{pmatrix} 1_g \\ M \end{pmatrix}$, and (b) M, which is automatically symmetric, has definite imaginary part, are both *open conditions*. □

Remark 3.20.
- *It follows that $\mathbb{H}_g^{\pm}$ is a Hermitian symmetric domain according to the definition given in section 2.5. The definition will be discussed later in more detail. The embedding $\mathbb{H}_g^{\pm} \hookrightarrow \{V^{-1,0} \subset \mathbb{C}^{2g}$ isotropic subspace$\}$ is a special case of the Borel embedding which exist for all Hermitian symmetric domains. We will later see the general construction.*
- *The complex structures induced on $\mathbb{H}_g^{\pm}$ by (9), and by the Borel embedding (12), respectively, agree. The structure is also determined by studying a moduli problem for families of complex tori. We will discuss this in the next section.*
- $\mathrm{GSp}(\phi_d, \mathbb{Z})$ *acts on* $\mathbb{H}_g^{\pm}$ *with* finite *stabilizer groups. These finite groups are precisely the groups of automorphisms of the corresponding* polarized *complex torus.*

3.21. As in the introduction, to get a moduli functor which is representable one is forced to introduce level structures. Let N be a positive integer coprime to all d_i. For a complex torus $X = \Lambda\backslash V$ one has obviously like for elliptic curves:

$$X[N] = \{x \in X \mid N \cdot x = 0\} \cong \Lambda\backslash \frac{1}{N}\Lambda \cong (\mathbb{Z}/N\mathbb{Z})^{2g}.$$

Define

$$\mathcal{A}_{g,d,N} := \left\{ \begin{array}{c} X, E \text{ as before} \\ \xi : (\mathbb{Z}/N\mathbb{Z})^{2g} \xrightarrow{\sim} X[N] \text{ symplectic similtude} \end{array} \right\}_{/\,\text{Iso.}}$$

On $\mathbb{Z}/N\mathbb{Z}$-modules we consider symplectic forms multiplicatively, for example, we have

$$\begin{array}{rcl} (\mathbb{Z}/N\mathbb{Z})^{2g} \times (\mathbb{Z}/N\mathbb{Z})^{2g} & \to & \mathbb{C}^* \\ v, w & \mapsto & \exp\left(\dfrac{2\pi i \phi_d(v,w)}{N}\right) \end{array} \tag{13}$$

and on

$$\begin{array}{rcl} X[N] \times X[N] = \Lambda\backslash \frac{1}{N}\Lambda \times \Lambda\backslash \frac{1}{N}\Lambda & \to & \mathbb{C}^* \\ v, w & \mapsto & \exp\left(N \cdot E(v,w)\right) \end{array} \tag{14}$$

where E is the polarization. This form is important and is called the **Weil pairing**.
ξ is supposed to be a symplectic similtude w.r.t. the forms (13) and (14). This condition is not necessary to have a good moduli problem but gives a finer information that will turn out to be convenient.
Note that if N is coprime to the d_i then these pairings are both perfect in the sense that they determine isomorphisms

$$(\mathbb{Z}/N\mathbb{Z})^{2g} \xrightarrow{\sim} \mathrm{Hom}((\mathbb{Z}/N\mathbb{Z})^{2g}, \mathbb{C}^*) \qquad X[N] \xrightarrow{\sim} \mathrm{Hom}(X[N], \mathbb{C}^*)$$

3.22. Adding the trivialization $\beta : \phi : H_1(X,\mathbb{Z}) \to \mathbb{Z}^{2g}$ (symplectic similitude) to the datum, we again get a simpler description:

$$\begin{aligned}
\mathbb{H}^{\pm}_{g,N} &:= \left\{ \begin{array}{c} X \text{ complex torus of dim } g \\ E \in \mathrm{NS}(X) \text{ a polarization of type } d \\ \phi : H_1(X,\mathbb{Z}) \to \mathbb{Z}^{2g} \\ \xi : (\mathbb{Z}/N\mathbb{Z})^{2g} \xrightarrow{\sim} X[N] \text{ symplectic similtude} \end{array} \right\}_{/\text{ Iso.}} \\
&\cong \left\{ \begin{array}{c} \text{complex structure on } V = \mathbb{R}^{2g} \\ \text{s.t. } \mathbb{Z}^{2g}\backslash V \text{ is polarized via } \phi_d \\ \xi : (\mathbb{Z}/N\mathbb{Z})^{2g} \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^{2g} \text{ symplectic similtude} \end{array} \right\} \\
&\cong \mathbb{H}^{\pm}_g \times \mathrm{GSp}(\phi_d, \mathbb{Z}/N\mathbb{Z})
\end{aligned}$$

and we have

$$\mathcal{A}_{g,d,N} = \mathrm{GSp}(\phi_d,\mathbb{Z})\backslash \left(\mathbb{H}^{\pm}_g \times \mathrm{GSp}(\phi_d, \mathbb{Z}/N\mathbb{Z})\right)$$

where $\mathrm{GSp}(\phi_d,\mathbb{Z})$ now acts on both factors from the left.

Lemma 3.23. *If $N \geq 3$ then $\mathrm{GSp}(\phi_d, \mathbb{Z}/N\mathbb{Z})$ is torsion free and acts freely and properly discontinuously on $\mathbb{H}^{\pm}_g \times \mathrm{GSp}(\phi_d, \mathbb{Z}/N\mathbb{Z})$*

It follows that $\mathcal{A}_{g,d,N}$ is a manifold w.r.t. the complex structure on $\mathbb{H}^{\pm}_g$ given by the descriptions (9) and (12). Now this structure is also determined because $\mathcal{A}_{g,d,N}$ does represent a moduli problem if $N \geq 3$:

Theorem 3.24. *If $N \geq 3$ and $(N, d_i) = 1$ then we have functorially in analytic manifolds S an isomorphism*

$$\mathrm{Hom}(S, \mathcal{A}_{g,d,N}) \cong \left\{ \begin{array}{c} \pi : X \to S \textit{ family of complex tori of dim } g \\ E \in R^2\pi_*\mathbb{Z}(1) \textit{ which is fibrewise a polarization of type } d \\ \xi : (\mathbb{Z}/N\mathbb{Z})^{2g}_S \xrightarrow{\sim} X[N] \textit{ symplectic similtude of } \mathbb{Z}/N\mathbb{Z}\textit{-local systems over } S \end{array} \right\}_{/\textit{ Iso.}}$$

Here, by a **family of complex tori** we understand a proper holomorphic submersion $X \to S$ of complex manifolds whose fibers are complex tori. We assume that their neutral elements assemble into a holomorphic section $e : S \to X$ or, equivalently, that $X \to S$ is a group object in the category of morphisms of complex manifolds $X \to S$.

Proof (Sketch). Given a family on the right hand side we get a classifying map

$$\varphi : S \to \mathcal{A}_{g,d,N}$$

by definition. The point is:

Why is φ holomorphic?

First note that locally on $U \subset B$, say, we get a lift of this map

$$\widetilde{\varphi} : U \to \mathbb{H}^{\pm}_g \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$$

by choosing a trivialization of local systems (symplectic similitude) $\widetilde{\beta} : R^1\pi_*\mathbb{Z}|_U \xrightarrow{\sim} \mathbb{Z}^{2g}_U$ (point-wise is gives the β of before).
The holomorphy of this map is best explained if we imagine the complex structure on $\mathbb{H}^{\pm}_g$ to be induced by the embedding (12)

$$\mathbb{H}^{\pm}_g \hookrightarrow M^\vee := \{V^{-1,0} \subset \mathbb{C}^{2g} \mid \text{isotropic subspace of dim } g\},$$

(isotropic, as usual, w.r.t. the form ϕ_d). The right hand side also represents a moduli problem (which determines its complex structure):

$$\mathrm{Hom}(S, M^\vee) = \left\{ \begin{array}{c} \textit{Holomorphic} \text{ subbundels } \mathcal{V}^{-1,0} \text{ of } (\mathcal{O}_S)^{2g} \\ \text{which are of dimension } g \text{ and (point-wise) isotropic} \end{array} \right\}_{/\text{ Iso.}}$$

Therefore the holomorphy of our map expresses the fact that the collection of the $V_b^{-1,0}$ for the various complex tori $X_b := \pi^{-1}(b)$ form a *homomorphic* subbundle of $\mathbb{C}_U^{2g}$. The space $V_b^{-1,0}$ is however the image under $\widetilde{\beta}_b$ of the subspace

$$H^0(X_b, \Omega^1_{X_b}) \subset H^1(X_b, \mathbb{C})$$

We claim that this comes from an embedding of holomorphic vector bundles

$$R^0\pi_*\Omega^1_{X/U} \hookrightarrow R^1\pi_*\mathbb{C}_X \otimes \mathcal{O}_U$$

where $\Omega^1_{X/U}$ is the sheaf of relative holomorphic differential form of degree 1. But for the right hand side we have the isomorphism of sheaves of Abelian groups (projection formula):

$$(R^1\pi_*\mathbb{C}_X) \otimes \mathcal{O}_U \cong R^1\pi_*\pi^{-1}\mathcal{O}_U$$

and we have the exact sequence of sheaves of Abelian groups

$$0 \longrightarrow \pi^{-1}\mathcal{O}_B \longrightarrow \mathcal{O}_X \xrightarrow{d} \Omega^{1,c}_{X|B} \longrightarrow 0$$

hence a map

$$R^0\pi_*\Omega^1_{X/U} = R^0\pi_*\Omega^{1,c}_{X/U} \to R^1\pi_*\pi^{-1}\mathcal{O}_U.$$

This defines a holomorphic sub*bundle*[7] because point-wise this map gives the embedding $H^0(X_b, \Omega^1_{X_b}) \subset H^1(X_b, \mathbb{C})$. □

3.2.5 More on polarizations

Let $X = \Lambda\backslash V$ be a complex torus. Recall that a subgroup of $H^1(X, \mathcal{O}_X^*)$ was given by

$$X^\vee = \mathrm{Hom}(\Lambda, \mathbb{Z}(1))\backslash H^1(X, \mathcal{O}).$$

This group is itself a complex torus. How can its analytic structure be characterized? Since $H^1(X, \mathcal{O}_X^*)$ sits in the exact sequence of Proposition 3.10 it gets a complex analytic structure by taking an infinite number of copies of $X^\vee$. We define $\mathrm{Pic}(X) := H^1(X, \mathcal{O}_X^*)$ with the so formed analytic structure.

Proposition 3.25. *We have functorially in analytic manifolds*

$$Hom(S, \mathrm{Pic}(X)) = \left\{ \begin{array}{c} \mathcal{L} \text{ line bundle on } X \times S \\ + \text{ trivialization } \beta : (0, \mathrm{id})^*\mathcal{L} \xrightarrow{\sim} \mathcal{O}_S \end{array} \right\}_{/\ Iso.}$$

Proof. We content ourselves by giving the universal line bundle on $X \times X^\vee$ (corresponding to the embedding $X^\vee \hookrightarrow \mathrm{Pic}(X)$ of the connected component of 1 on the left hand side). To describe it, we have to give a cocycle in

$$H^1(X \times X^\vee, \mathcal{O}_X^*) \cong H^1(\Lambda \oplus \mathrm{Hom}(\Lambda, \mathbb{Z}(1)), \mathcal{O}^*(V \oplus V^\vee)).$$

where $V^\vee = H^1(X, \mathcal{O}) = \mathrm{Hom}(\Lambda, \mathbb{Z}(1)) \otimes \mathbb{R} = \overline{\mathrm{Hom}(V, \mathbb{C})}$. This can be explicitly described by the canonical cocycle

$$\begin{array}{rcl} \Lambda \oplus \mathrm{Hom}(\Lambda, \mathbb{Z}(1)) & \to & \mathcal{O}^*(V \oplus V^\vee) \\ (\sigma, f) & \mapsto & \exp\left(z^\vee(\sigma) - f(z)\right) \end{array} \tag{15}$$

δ of it is *twice* the canonical symplectic form

$$\begin{array}{r} \Lambda \oplus \mathrm{Hom}(\Lambda, \mathbb{Z}(1)) \times \Lambda \oplus \mathrm{Hom}(\Lambda, \mathbb{Z}(1)) \to \mathbb{Z}(1) \\ (\lambda_1, f_1), (\lambda_2, f_2) \mapsto f_2(\lambda_1) - f_1(\lambda_2) \end{array}$$

It is called the **Poincaré line bundle**. □

[7]Note that, in general, a holomorphic subsheaf of a locally free subsheaf need not to define an embedding of bundles.

The Poincaré bundle $\mathcal{P}$ on $X \times X^\vee$ has the property that there is also a canonical trivialization

$$\beta' : (\mathrm{id}, 0)^* \mathcal{L} \xrightarrow{\sim} \mathcal{O}_{X^\vee}$$

Interchanging the rôles of X and $X^\vee$ we therefore get a canonical morphism

$$\mathrm{can.} \in \mathrm{Hom}(X, (X^\vee)^\vee).$$

Note also that $X \mapsto X^\vee$ is a contravariant endo-functor of the category of complex tori. Because of this is makes sense to speak about symmetric morphisms $\rho : X \to X^\vee$, i.e. those where $\rho^\vee = \rho$, i.e. such that the diagram

$$\begin{tikzcd} X \arrow[r, "\rho"] \arrow[d, "\mathrm{can.}"'] & X^\vee \arrow[d, equal] \\ (X^\vee)^\vee \arrow[r, "\rho^\vee"] & X^\vee \end{tikzcd}$$

commutes.

Lemma 3.26.

$$\mathrm{NS}(X) = \{\rho : \mathrm{Hom}(X, X^\vee) \mid \rho^\vee = \rho\},$$

where the Hom *has to be taken in the category of complex tori.*

Proof. We have seen in the very beginning (cf. 3.1) that a morphism $\rho : X \to X^\vee$ corresponds to a morphism

$$\Lambda \to \mathrm{Hom}(\Lambda, \mathbb{Z}(1))$$

which respects the corresponding complex structures, i.e. Hodge structures of weight -1 on these spaces. The Hodge structure on Λ is the one describing the complex torus as $X = \Lambda \backslash V$, where $V = \Lambda \otimes \mathbb{R}$. The Hodge structure on $\mathrm{Hom}(\Lambda, \mathbb{Z}(1))$ is the one coming from the embedding $\mathrm{Hom}(\Lambda, \mathbb{Z}(1)) \hookrightarrow H^1(X, \mathcal{O})$. The corresponding Hodge decomposition of the latter is

$$\mathrm{Hom}(\Lambda, \mathbb{Z}(1)) \otimes \mathbb{C} = \mathrm{Hom}(\Lambda, \mathbb{C}) \cong H^1(X, \mathbb{C}) \cong H^1(X, \mathcal{O}) \oplus H^0(X, \Omega)$$

which was dual to the one on Λ, except that the weight has to be -1 instead of 1. This is elegantly resolved by defining a Hodge structure on $\mathbb{Z}(1)$ of weight -2 setting $\mathbb{C}^{-1,-1} = \mathbb{C}(1) = \mathbb{C}$ [8]. Then the complex structure on $\mathrm{Hom}(\Lambda, \mathbb{Z}(1))$ is described by the Hodge structure (now of weight -1)

$$\mathrm{Hom}(\Lambda, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}(1)$$

where all operations Hom, $\otimes$, etc. are understood to be operations of Hodge structures. We get identifications of Hodge structures:

$$\mathrm{Hom}(\Lambda \otimes \Lambda, \mathbb{Z})^{1,1} = \mathrm{Hom}(\Lambda \otimes \Lambda, \mathbb{Z}(1))^{0,0} = \mathrm{Hom}(\Lambda, \mathrm{Hom}(\Lambda, \mathbb{Z}(1)))^{0,0}$$

where the last group is precisely the group of morphisms of Hodge structures. Hence we have

$$\mathrm{Hom}(\Lambda \otimes \Lambda, \mathbb{Z})^{1,1} \cong \mathrm{Hom}(X, X^\vee). \tag{16}$$

Because of the minus sign in (15) the morphism $\mathrm{can.} : X \to (X^\vee)^\vee$ is given on the level of lattices by minus the canonical morphism

$$\Lambda \to \mathrm{Hom}(\mathrm{Hom}(\Lambda, \mathbb{Z}(1)), \mathbb{Z}(1)).$$

Hence in (16) skew symmetric forms correspond to symmetric morphisms. □

[8] This also could be achieved by the defining $\mathbb{Z}(1) = H_2(\mathbb{P}^2(\mathbb{C}), \mathbb{Z})$. Note that orientations of $\mathbb{P}^2(\mathbb{C})$ also correspond naturally to choices of $\sqrt{-1}$ in $\mathbb{C}$, thus $H_2(\mathbb{P}^2(\mathbb{C}), \mathbb{Z}) \cong 2\pi i\mathbb{Z}$ as Abelian group.

3.3 Algebraic theory

3.3.1 Abelian varieties

Let $\overline{k}$ be an algebraically closed field. In analogy with the analytic definition of a complex torus, we define:

Definition 3.27. *An* **Abelian variety** $\pi : A \to S$ *of dimension* g *is a connected proper group variety, defined over* k*, of dimension* g.

In analogy with the complex theory, we have:

Proposition 3.28.

1. *An Abelian variety is smooth and commutative.*
2. *Let* A, B *be Abelian varieties. Any morphism of varieties (not group varieties)* $A \to B$ *is of the form*

$$x \mapsto h(x) + a$$

for a point $a \in A(\overline{k})$ *and a homomorphism of group varieties* $h : A \to B$.

If $\overline{k}$ is a subfield of $\mathbb{C}$, we can speak about the $\mathbb{C}$-valued points $A(\mathbb{C})$ of an Abelian variety A. It is a compact connected complex Lie group, i.e. a complex torus.
We will also need Abelian varieties that are defined over a field k which is not algebraically closed, and also families of Abelian varieties parametrized by varieties.

> *To understand this technically precisely one should learn about schemes. We will avoid talking about schemes for pedagogical reasons to the disadvantage that all statements will remain a bit vague. More or less everything should be literally true if one understands:*
>
> *"variety defined over a field* k *= reduced scheme of finite type over* $\mathrm{spec}(k)$*".*

Let S be a variety defined over some field k.

Definition 3.29. *A family of Abelian varieties* $\pi : A \to S$ *of relative dimension* g *is a family of group varieties*[9] *over* S*, s.t.*

1. π *is proper and smooth,*
2. *for all points* $p \in S(K)$*, where* K *is an algebraically closed field containing* k*, the fiber* X_p *(which is defined over* K*) is connected of dimension* g.

If k is a subfield of $\mathbb{C}$ and S is itself smooth, then $A(\mathbb{C}) \to S(\mathbb{C})$ is a family of complex tori in the sense of the previous section.

3.3.2 Line bundles and polarizations

Our goal is an algebraic description of the moduli problem represented by $\mathcal{A}_{g,d,N}$ thus producing an algebraic model of it. The definition of $\mathcal{A}_{g,d,N}$ involved the notion of polarization. How can it be described algebraically? Let A be an Abelian variety defined over $\mathbb{C}$. Recall that polarization on the complex torus $X := A(\mathbb{C})$ was defined as an element of $\mathrm{NS}(X)$ satisfying some kind of positivity. We obtained the space $\mathrm{NS}(X)$ as a quotient of $H^1(X, \mathcal{O}^*)$, the group parametrizing line bundles on X. Now we have the following two facts:

1. Every complex vector bundle on $X = A(\mathbb{C})$ is automatically algebraic.
2. Let $\mathcal{L}$ be a complex (equivalently: algebraic) line bundle on A. Then $\mathcal{L}$ is ample if and only if $\delta([\mathcal{L}]) \in \mathrm{NS}(X)$ is a polarization.

[9] i.e. a group object in the category of morphisms of varieties $X \to S$

Fact 2. is just Riemann's theorem 3.12. A reason for fact 1. will be explained later. It means in particular that the group of isomorphism classes of algebraic, resp. of analytic line bundles are isomorphic (the former group is called the **Picard group** of the variety):

$$\mathrm{Pic}(A) := H^1_{\mathrm{Zar}}(A, \mathcal{O}_A^*) \cong H^1(X, \mathcal{O}_X^*).$$

Analytically we had the exact sequence

$$0 \longrightarrow X^\vee \longrightarrow H^1(X, \mathcal{O}_X^*) \longrightarrow \mathrm{NS(X)} \longrightarrow 0.$$

It was mentioned already that $A^\vee = \mathrm{Hom}(\Lambda, \mathbb{Z}(1))/H^1(X, \mathcal{O})$ is again a complex torus hence it is reasonable to expect that the condition for a line bundle $\mathcal{L}$ to be in $X^\vee$ has an algebraic description such that $X^\vee = A^\vee(\mathbb{C})$ for some complex torus that represents a moduli problem for them. This is indeed the case, and the corresponding subgroup will be called $\mathrm{Pic}^0(A)$. We hence get our algebraic description:

Definition 3.30. *Let A be an Abelian variety defined over $\overline{k}$. Then we define*

$$\mathrm{NS}(A) := \mathrm{Pic}(A)/\mathrm{Pic}^0(A).$$

A **polarization** *is a class $[\mathcal{L}]$ in $\mathrm{NS}(A)$ such that $[\mathcal{L}]^{\otimes \pm 1}$ consists of ample line bundles.*

3.31. For families of Abelian varieties, or Abelian varieties defined over an arbitrary field, this description is not completely appropriate. One would have to give meaning to the field of definition of a class $[\mathcal{L}]$ or to a corresponding notion in families. This can be done but it is easier to mimic the description that we got in section 3.2.5:

$$\mathrm{NS}(X) = \{\rho : \mathrm{Hom}(X, X^\vee) \mid \rho^\vee = \rho\}.$$

If we are able to describe the complex torus $X^\vee$ algebraically then we obtain a different description where it makes sense to speak about a field of definition of the objects, for example.

We get analogously to Proposition 3.25:

Theorem 3.32. *Let $\pi : A \to S$ be a family of Abelian varieties. There exists a group variety $\mathrm{Pic}(A/S)$ s.t.*

$$Hom_S(T, \mathrm{Pic}(A/S)) = \left\{ \begin{array}{c} \mathcal{L} \textit{ line bundle on } X \times_S T \\ + \textit{ trivialization } \beta : (0, \mathrm{id})^* \mathcal{L} \xrightarrow{\sim} \mathcal{O}_T \end{array} \right\}_{/\ Iso.}$$

and a closed subgroup variety $\mathrm{Pic}^0(A/S)$ over S, s.t. for $S = \{p\}$ a point over an algebraically closed fields:

$$\mathrm{Pic}^0(A_p)(\overline{k}) = \left\{ \begin{array}{c} [\mathcal{L}] \in \mathrm{Pic}(A_p)(\overline{k}) \\ \textit{s.t. } \mathcal{L} \textit{ is algebraically equivalent to 0} \end{array} \right\}.$$

The morphism $\mathrm{Pic}^0(A/S) \to S$ is a family of Abelian varieties, denoted $A^\vee$.

Two line bundles $\mathcal{L}_1, \mathcal{L}_2$ on a variety X, defined over $\overline{k}$, are called **algebraically equivalent** if there exists a connected variety V, defined over $\overline{k}$, points $v_1, v_2 \in V(\overline{k})$, and a line bundle $\mathcal{L}$ on $X \times V$ such that $(v_1, \mathrm{id})^* \mathcal{L} \cong \mathcal{L}_1$ and $(v_2, \mathrm{id})^* \mathcal{L} \cong \mathcal{L}_2$.
For an Abelian variety A, defined over $\mathbb{C}$, we have by definition that $\mathrm{Pic}^0(A/\mathbb{C})$ is the connected component of zero in $\mathrm{Pic}(A/\mathbb{C})$. Since algebraic and analytic line bundles are the same, $\mathrm{Pic}^0(A/\mathbb{C})$ is actually the connected component and isomorphic to $A(\mathbb{C})^\vee$.
Let

$$m : A \times_S A \to A$$

be the group law and let $\mathcal{L}$ be a line bundle on A. Then

$$m^*(\mathcal{L}) \otimes \mathrm{pr}_1^*(\mathcal{L})^{-1} \otimes \mathrm{pr}_2^*(\mathcal{L})^{-1}$$

defines a line bundle on $A \times_S A$ and therefore determines a morphism

$$\Lambda(\mathcal{L}) : A \to A^\vee$$

If A is defined over an algebraically closed field $\overline{k}$ then on $\overline{k}$-points this map is given by

$$\begin{array}{rcl} \Lambda(\mathcal{L})(\overline{k}) : A(\overline{k}) & \to & A^\vee(\overline{k}) \\ a & \mapsto & T_a^*\mathcal{L} \otimes \mathcal{L}^{\otimes -1} \end{array} \tag{17}$$

where $T_a : A \to A$ is the map 'translation by a'.
Idea of construction of $\mathrm{Pic}(A/S)$: Prove that $\Lambda(\mathcal{L})(\overline{k})$ is a group homomorphism (which is clear, once we know that $A^\vee = \mathrm{Pic}^0(A/S)$ exists as an Abelian variety) with some finite kernel K and define $A^\vee := A/K$ as the quotient of group varieties. The fact that this is a group homomorphism is called the theorem of the square and is a consequence of the following general

Theorem 3.33 (Theorem of the cube)**.** *Let X, Y, Z be complete connected varieties over a field with choosen base points x_0, y_0, and z_0. If $\mathcal{L}$ is a line bundle on $X \times Y \times Z$ whose restriction to the subvarieties*

$$\{x_0\} \times Y \times Z \qquad X \times \{y_0\} \times Z \qquad X \times Y \times \{z_0\}$$

is trivial then also $\mathcal{L}$ is trivial.

Corollary 3.34 (Theorem of the square)**.** *We have*

$$T_{a+b}^*\mathcal{L} \otimes \mathcal{L}^{\otimes -1} \cong T_a^*\mathcal{L} \otimes \mathcal{L}^{\otimes -1} \otimes T_b^*\mathcal{L} \otimes \mathcal{L}^{\otimes -1}.$$

In particular, the map (17) is a group homomorphism.

Proof. Apply the theorem of the cube to the line bundle

$$m_{123}^*\mathcal{L} \otimes (m_{12}^*\mathcal{L})^{\otimes -1} \otimes (m_{23}^*\mathcal{L})^{\otimes -1} \otimes (m_{31}^*\mathcal{L})^{\otimes -1} \otimes p_1^*\mathcal{L} \otimes p_2^*\mathcal{L} \otimes p_3^*\mathcal{L} \tag{18}$$

on A^3. Here $m_{ij} = p_i + p_j$ and $m_{123} = p_1 + p_2 + p_3$. Then pull-back this bundle, which is thus trivial, along the map $A \to A^3$ given by $x \mapsto (x, a, b)$. $\square$

Corollary 3.35. *Let $n \in \mathbb{Z}$ and let $[n] : A \to A$ be the multiplication by n map. We have*

$$[n]^*\mathcal{L} \cong \mathcal{L}^{\otimes \frac{n^2+n}{2}} \otimes ([-1]^*\mathcal{L})^{\otimes \frac{n^2-n}{2}}.$$

Proof. The formula is obviously true for $n = -1, 0, 1$. Consider the line bundle (18) again, pull it back via the map $A \to A^3$ given by $x \mapsto ([n+1]x, x, -x)$ and use induction by n (in both directions). $\square$

In particular for **symmetric** line bundles, i.e. those with $[-1]^*\mathcal{L} \cong \mathcal{L}$, we have that $[n]^*\mathcal{L} \cong \mathcal{L}^{\otimes n^2}$ and for **skew-symmetric** line bundles, i.e. those with $[-1]^*\mathcal{L} \cong \mathcal{L}^{\otimes -1}$ we have that $[n]^*\mathcal{L} \cong \mathcal{L}^{\otimes n}$. Because of the equation

$$\mathcal{L}^{\otimes 2} \cong \underbrace{(\mathcal{L} \otimes [-1]^*\mathcal{L})}_{\text{symmetric}} \otimes \underbrace{(\mathcal{L} \otimes ([-1]^*\mathcal{L})^{\otimes -1})}_{\text{skew-symmetric}}$$

we see that up to inverting 2 we get a decompositon of line-bundles into a symmetric and skew-symmetric one. This reflects the analytic fact that $\delta : H^1(X, \mathcal{O}_X^*) \to \mathrm{NS}(X)$ had a canonical section $\frac{1}{2}s$ (cf. 3.11) after inverting 2, in the following way:
The map $\Lambda(\mathcal{L})$ is equal to the analytic map δ, i.e. for $k = \mathbb{C}$ there is a map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & X^\vee \arrow[r] \arrow[d, "\sim"] & H^1(X, \mathcal{O}_X^*) \arrow[r, "\delta"] \arrow[d, "\sim"] & \mathrm{NS}(X) \arrow[r] \arrow[l, bend left, "s"] \arrow[d, hook, "\iota"] & 0 \\
0 \arrow[r] & A^\vee(\mathbb{C}) \arrow[r] & \mathrm{Pic}(A)(\mathbb{C}) \arrow[r, "\Lambda"] & \mathrm{Hom}(A, A^\vee) \arrow[l, bend left, "s"] &
\end{tikzcd}$$

The algebraic section s is given by the map

$$\rho \mapsto (\mathrm{id}, \rho)^*\mathcal{P}$$

where $\mathcal{P}$ is the universal line bundle on $A \times A^\vee$. One can show that $(\mathrm{id}, \rho)^*\mathcal{P}$ is automatically symmetric. To make sense of this, we have to establish algebraically that $(A^\vee)^\vee \cong A$, see 3.36 below.
Furthermore an application of the definitions and Corollary 3.35 shows that

$$s(\Lambda(\mathcal{L})) = \mathcal{L} \otimes [-1]^*\mathcal{L} \qquad \Lambda(s(E)) = E + E^\vee \ (= 2E \text{ if } E \text{ is symmetric}).$$

Because $\mathrm{Hom}(A, A^\vee)$ is torsion-free, s is injective and in particular, $\mathcal{L}$ is skew-symmetric if and only if $\Lambda(\mathcal{L}) = 0$.
We can summarize by saying that the following are equivalent conditions for a line bundle $\mathcal{L}$ on A:

1. skew-symmetric,
2. translation-invariant,
3. $\Lambda(\mathcal{L})$ is the zero-homomorphism,
4. algebraically equivalent to 0, i.e. $\in \mathrm{Pic}^0(A)$.

3.36. Let now A be an Abelian variety over any field k. Also in the algebraic situation the universal line bundle $\mathcal{P}$ on $A^\vee \times A$ is trivial on $A^\vee \times \{0\}$ (by definition of Pic) and trivial on $\{0\} \times A$ (because this is the line bundle on A parametrized by the morphism $0 : \{\cdot\} \to A^\vee$). Hence this gives a morphism

$$A \to \mathrm{Pic}(A^\vee)$$

which can be shown to induce an isomorphism

$$A \to (A^\vee)^\vee.$$

It thus makes sense to define

Definition 3.37. *A morphism $\rho : A \to A^\vee$ is called* **symmetric** *if $\rho^\vee = \rho$ via the identification above. We define*

$$NS(A) := \{\rho \in \mathrm{Hom}(A, A^\vee) \mid \rho^\vee = \rho\}.$$

This even makes sense in a family $A \to S$ insisting that $\rho^\vee = \rho$ holds for any point of S defined over an algebraically closed field.

One can show purely algebraically that the image of the morphism Λ has values in symmetric morphisms.

Definition 3.38. *A homomorphism $\rho \in \mathrm{NS}(A)$ is called a* **polarization** *if $\pm\rho$ is of the form $\Lambda(\mathcal{L})$ for an* ample *line bundle $\mathcal{L}$ defined over an algebraically closed field. This again makes sense even in families.*

To get an algebraic version of the moduli problem discussed in the analytic setting, we have to discuss level structures and an algebraic analogue of the Weil pairing.
First we have

Proposition 3.39. *For any isogeny $\alpha : A \to B$ between Abelian varieties over S there exists a non-degenerate pairing*

$$\ker(\alpha) \times \ker(\alpha^\vee) \to \mathbb{G}_m.$$

Here $\ker(\alpha)$, and $\ker(\alpha^\vee)$, respectively, are considered to be (families of) finite group varieties[10]. If S is a point defined over an algebraically closed field of characteristic 0 this is the same as considering them as sets.

[10] If S does not live in characteristic 0 then these group 'varieties' might not be reduced and so one really should work with schemes.

Proof. We restrict to the case, where A is defined over an algebraically closed field (of characteristic zero). We have the following sequence of group varieties:

$$0 \longrightarrow \ker(\alpha) \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0$$

and from it, we get a long exact sequence of Ext-groups associated with it:

$$\mathrm{Hom}(A, \mathbb{G}_m) \longrightarrow \mathrm{Hom}(\ker(\alpha), \mathbb{G}_m) \xrightarrow{\delta} \mathrm{Ext}^1(B, \mathbb{G}_m) \longrightarrow \mathrm{Ext}^1(A, \mathbb{G}_m) \tag{19}$$

Instead of referring to any abstract theory here (one can do this by considering the sequence in the category of Abelian f.p.p.f. sheaves, for instance), we explicitly have:

$$\mathrm{Ext}^1(A, \mathbb{G}_m) = \left\{ 1 \longrightarrow \mathbb{G}_m \longrightarrow \widetilde{A} \longrightarrow A \longrightarrow 1 \right\}_{/\mathrm{Iso.}}$$

where the sequences are extensions of group varieties. The connecting homomorphism δ in the sequence (19) is given by mapping a homorphism $\beta : \ker(\alpha) \to \mathbb{G}_m$ to the quotient of the trivial extension $A \times \mathbb{G}_m$, where $\ker(\alpha)$ acts on A by translation and on $\mathbb{G}_m$ via β. One can show that every extension $\widetilde{A}$ in $\mathrm{Ext}^1(A, \mathbb{G}_m)$ is always Zariski-locally trivial and this give rise to a line bundle on A by considering the same cocycle in $H^1_{\mathrm{Zar}}(A, \mathbb{G}_m) = H^1_{\mathrm{Zar}}(A, \mathcal{O}_A^*)$. This means more concretely that:

$$\widetilde{A} = \text{total space of the line bundle with zero section removed.}$$

Conversely for every line bundle $\mathcal{L}$ in $\mathrm{Pic}^0(A)$ with fixed trivialization of $\mathcal{L}_0$, the bundle

$$m^*\mathcal{L} \otimes (p_1^*\mathcal{L})^{\otimes -1} \otimes (p_2^*\mathcal{L})^{\otimes -1}$$

is trivial because it defines the morphism $0 = \Lambda(\mathcal{L}) : A \to A^\vee$. In particular, we have a canonical identification

$$\mathcal{L}_a \otimes \mathcal{L}_b \cong \mathcal{L}_{a+b}$$

which allows to define a group structure on the total space of the line bundle minus the zero section. The associativity of this group structure follows from a careful analysis of the proof of the theorem of the square. This gives an isomorphism

$$\mathrm{Pic}^0(A) \cong \mathrm{Ext}^1(A, \mathbb{G}_m).$$

Hence the statement of the Proposition follows from the exactness of sequence (19).
There is an alternative, more elementary description of the duality

$$\ker(\alpha) \times \ker(\alpha^\vee) \to \mathbb{G}_m.$$

as follows: Let $\mathcal{L}$ be a line bundle in $\mathrm{Pic}^0(B)$ with trivialization $\beta : \mathcal{L}_0 \cong \overline{k}$. Asume that there is an isomorphism $\mu : \alpha^*\mathcal{L} \cong \mathcal{O}_A$ which we may choose compatible with β. But looking at the sequence

$$0 \longrightarrow \ker(\alpha) \xrightarrow{\iota} A \xrightarrow{\alpha} B \longrightarrow 0$$

we see that $\alpha^*\mathcal{L}|_{\ker(\alpha)}$ has a different trivialization namely the composition $\widetilde{\beta}$ of the canonical identification

$$\iota^*\alpha^*\mathcal{L} = 0^*\mathcal{L} \otimes_k \mathcal{O}_{\ker(\alpha)}$$

with β. The comparison $\mu \circ \widetilde{\beta}^{-1}$ is an element in $\mathcal{O}^*_{\ker(\alpha)} = \mathrm{Hom}(\ker(\alpha), \mathbb{G}_m)$. One shows that this describes the same pairing as the abstract one. $\square$

3.40. Hence, given a polarization ρ, we can define a pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle_\rho : A[N] \times A[N] &\to \mathbb{G}_m \\ x, y &\mapsto \langle x, \rho(y) \rangle \end{aligned}$$

This pairing is called the **Weil pairing**. It is non-degenerate as soon as ρ induces an isomorphism $A[N] \cong A^\vee[N]$ which is the case if and only if the degree of ρ is coprime to N.

Exercise 3.41. *Show that this pairing is skew-symmetric and coincides with the analytically defined Weil pairing for $X = A(\mathbb{C})$ if A is defined over $\mathbb{C}$.*

3.3.3 The Tate module

Theorem 3.42. *If A is an Abelian variety defined over an algebraically closed field of characteristic prime to some $N \in \mathbb{N}$ then*

$$A[N] \cong (\mathbb{Z}/N\mathbb{Z})^{2g}$$

where g is the dimension of A.

Corollary 3.43.

1. *If A is an Abelian variety defined over an algebraically closed field $\overline{k}$ and If l is a prime not equal to the characteristic of K then*

$$T_l(A) := \lim_{\leftarrow n} A[l^n] \cong (\mathbb{Z}_l)^{2g}.$$

If A is defined over a subfield $k \subset \overline{k}$ then $T_l(A_{\overline{k}})$ carries a continuous action of the Galois group $\mathrm{Gal}(\overline{k}/k)$. *We have a perfect pairing*

$$T_l(A_{\overline{k}}) \times T_l(A^\vee_{\overline{k}}) \to T_l(\mathbb{G}_{m,\overline{k}}) =: \mathbb{Z}_l(1)$$

of continuous $\mathrm{Gal}(\overline{k}/k)$*-modules.*

2. *If k is of characteristic zero, we define similarly*

$$T(A_{\overline{k}}) := \lim_{\leftarrow N} A_{\overline{k}}[N] \cong \widehat{\mathbb{Z}}^{2g}.$$

3. *A polarization E induces an alternating form*

$$T(A_{\overline{k}}) \times T(A_{\overline{k}}) \to T(\mathbb{G}_{m,\overline{k}}) =: \mathbb{Z}(1),$$

which is $\mathrm{Gal}(\overline{k}/k)$ *equivariant, if E is defined over k as well.*

4. *(Snake lemma) For an exact sequence*

$$0 \longrightarrow \ker(\alpha) \xrightarrow{\iota} A \xrightarrow{\alpha} B \longrightarrow 0$$

the sequence

$$0 \longrightarrow T(A_{\overline{k}}) \xrightarrow{T(\alpha)} T(B_{\overline{k}}) \longrightarrow \ker(\alpha)(\overline{k}) \longrightarrow 0$$

is exact.

5. *For $k = \mathbb{C}$ with $\Lambda\backslash V = A(\mathbb{C})$ we have canonically $\Lambda \otimes \widehat{\mathbb{Z}} \cong T(A)$ and for a polarization E we have get a commutative diagram*

$$\begin{tikzcd}
E : \Lambda \otimes \widehat{\mathbb{Z}} \arrow[d, "\sim"] & \times & \Lambda \otimes \widehat{\mathbb{Z}} \arrow[r] \arrow[d, "\sim"] & \mathbb{Z}(1) \otimes \widehat{\mathbb{Z}} \arrow[d, "\sim"] \\
\langle \cdot, \cdot \rangle_\rho : T(A) & \times & T(A) \arrow[r] & \widehat{\mathbb{Z}}(1)
\end{tikzcd}$$

where ρ is the algebraic polarization corresponding to E. Note that the exponential function induces a canonical isomorphism

$$\exp : \mathbb{Z}(1) \otimes \widehat{\mathbb{Z}} \to \widehat{\mathbb{Z}}(1)$$

In particular an analytic polarization is of type $d = (d_1, \ldots, d_g)$ if and only if the corresponding alternating form $\langle \cdot, \cdot \rangle_\rho$ on $T(A)$ is of type $d = (d_1, \ldots, d_g)$. In view of 4. of the corollary this is also equivalent to $\ker(\rho) \cong (\mathbb{Z}/d_1\mathbb{Z})^2 \times \cdots \times (\mathbb{Z}/d_g\mathbb{Z})^2$. This motivates the definition:

Definition 3.44. *Let A be an Abelian variety defined over a field k. A polarization ρ is called of type $d = (d_1, \dots, d_g)$ where $d_i \in \mathbb{N}$ and $d_1|d_2|\cdots|d_g$ if*

$$\ker(\rho) \cong (\mathbb{Z}/d_1\mathbb{Z})^2 \times \cdots \times (\mathbb{Z}/d_g\mathbb{Z})^2,$$

or, equivalently, if the Weil pairing on $T(A)$ is of type d.

This definition does only make sense if the characteristic of the field k is coprime to d. It can be extended to families $A \to S$ requiring the property for all points of S defined over algebraically closed fields.

Proof (sketch) of Theorem 3.42. We sketch the proof of the Theorem using a bit of intersection theory. If D is an ample divisor on an algebraic variety V, then the degree of a finite map

$$\varphi : V \to V'$$

might be computed as

$$\deg(\varphi) = \frac{(\varphi^* D)^{\dim(V)}}{D^{\dim(V)}}$$

where $D^{\dim(V)}$ refers to the highest intersection number. Now take $V = A$ and let $\mathcal{L}$ be an ample line bundle. Then $\mathcal{L} \otimes ([-1]^*\mathcal{L})$ is ample and symmetric, hence we may assume w.l.o.g. that $\mathcal{L}$ is ample and symmetric. In this case we have seen (cf. 3.35) that $[N]^*\mathcal{L} \cong \mathcal{L}^{\otimes N^2}$. From this we can infer that the kernel of $[N]$ must be finite because a morphism which send a positive dimensional subvariety to a point cannot have the property of preserving ampleness of a line bundle. Therefore $[N]$ is a finite morphism and for the divisor D associated with $\mathcal{L}$, we get $[N]^*D \sim N^2 D$ and hence

$$\deg([N]) = N^{2g}.$$

Since N does not divide the characteristic of $\overline{k}$, we can infer from this that

$$\#\ker([N])(\overline{k}) = N^{2g}.$$

This does not determine the structure of $\ker([N])$ yet but since this holds for any N, a combinatorial argument together with

$$\ker([M])(\overline{k}) = \{X \in \ker([N])(\overline{k}) \mid M \cdot X = 0\}$$

for $M|N$, shows that

$$\ker([N])(\overline{k}) \cong (\mathbb{Z}/N\mathbb{Z})^{2g}.$$

$\square$

There is also an algebraic analogue of Riemann's theorem which follows from the Theorem of Riemann-Roch, and Mumford vanishing, respectively.

Theorem 3.45. *For $\mathcal{L} = \mathcal{O}_A(D)$ a line bundle with associated divisor D we have*

1. *Riemann-Roch:*
$$\chi(\mathcal{L}) := \sum_i (-1)^i \dim H^i(A, \mathcal{L}) = \frac{D^g}{g!}.$$

2. *Mumford vanishing: If $\ker(\Lambda(\mathcal{L}))$ is finite then there exists an integer $i(\mathcal{L})$ with the property*
$$H^i(A, \mathcal{L}) = \{0\} \quad \textit{for } i \neq i(\mathcal{L})$$
and we have $i(\mathcal{L}) = 0$ if and only if $\mathcal{L}$ is ample.

Interestingly, in the analytic case, $i(\mathcal{L}) = 0$ is precisely the signature of H, the Hermitian form associated with $E = \delta(\mathcal{L})$. Note that in this case $\Lambda(\mathcal{L}) = E$ and so the assumption translated into: E non-degenerate on $H_1(X, \mathbb{R})$. From the theorem one can infer

Corollary 3.46. *Let $\mathcal{L}$ be an ample line bundle on an Abelian variety A, then*

1. $\dim H^0(X, \mathcal{L}) = \sqrt{\#\ker(\Lambda(\mathcal{L}))}$,
2. *$\mathcal{L}^{\otimes 3}$ is very ample.*

3.3.4 Moduli

The purpose of this section is to explain the following main theorem. The ingredients of the statement have been discussed in detail in the previous sections.

Main theorem 3.47 (Mumford)**.** *Let $N \geq 3$ be an integer and $d = (d_1, \ldots, d_g)$ a type, comprime to N. Then there exists a smooth quasi-projective variety $\mathcal{A}^{\mathrm{alg}}_{g,d,N}$, defined over $\mathbb{Q}$, such that for each variety S, defined over $\mathbb{Q}$,*

$$\mathrm{Hom}(S, \mathcal{A}^{\mathrm{alg}}_{g,d,N}) \cong \left\{ \begin{array}{c} \textit{A family of Abelian varieties over } S \textit{ of dimension } g \\ E \in \mathrm{NS}(A) \textit{ a polarization of type } d \\ \xi : (\mathbb{Z}/N\mathbb{Z})^{2g}_S \xrightarrow{\sim} A[N] \textit{ symplectic similitude} \end{array} \right\}_{/\ \mathit{Iso.}}$$

Here, the isomorphisms have to respect the respective polarizations up to sign only.

Corollary 3.48. *There is a canonical biholomorphic isomorphism*

$$\mathcal{A}^{\mathrm{alg}}_{g,d,N}(\mathbb{C}) \cong \mathcal{A}_{g,d,N}$$

In particular the Shimura variety $\mathcal{A}_{g,d,N}$ has a natural model defined over $\mathbb{Q}$. In the next section, we will investigate how this model can be characterized uniquely in a more group theoretical way using Deligne's notion of canonical model.

Proof of Corollary 3.48. The universal family of Abelian varieties

$$A^{\mathrm{univ}} \to \mathcal{A}^{\mathrm{alg}}_{g,d,N}$$

gives rise to a family of complex tori $X := A^{\mathrm{univ}}(\mathbb{C})$

$$\pi : X \to \mathcal{A}^{\mathrm{alg}}_{g,d,N}(\mathbb{C}).$$

It is equipped with a family of polarizations $E \in \mathrm{Hom}(A^{\mathrm{univ}}, (A^{\mathrm{univ}})^{\vee})$ which translates into an analytic family of polarizations $E \in R^2\pi_*\mathbb{Z}(1)$. The level structure also translates and hence X is parametrized by a holomorphic morphism

$$\mathcal{A}^{\mathrm{alg}}_{g,d,N}(\mathbb{C}) \to \mathcal{A}_{g,d,N}.$$

Furthermore it is a bijection of sets because Abelian varieties and complex tori over $\mathbb{C}$ form equivalent categories and the notions of polarizations and level-structures correspond, as we have seen. A bijective holomorphic morphism between analytic manifolds is automatically a biholomorphic isomorphism. □

Idea of proof of Theorem 3.47. In the sequel we write d, by abuse of notation, also for the product $d_1 \cdots d_g$. If E is a polarization, it is (at least point-wise over an algebraically closed field) by definition of the form $\Lambda(\mathcal{L})$ for some ample line bundle $\mathcal{L}$. We have the section s of 2Λ defined by $s(E) = (\mathrm{id}, E)^*\mathcal{P}$. We know that $s(E) = s(\Lambda(\mathcal{L})) = \mathcal{L} \otimes [-1]^*\mathcal{L}$. It follows that $s(E)$ is ample as well. By Corollary 3.46 $s(E)^{\otimes 3}$ is a very ample line bundle, which by construction depends only on the polarization. By the corollary $M_{A,E} := \pi_* s(E)^{\otimes 3}$ is a vector bundle of rank $m + 1 := 6^g d$ on S. Hence we can define a functor

$$\mathcal{A}^{\mathrm{rig}}_{g,d,N}(S) \cong \left\{ \begin{array}{c} A, E, \xi \text{ as before} \\ +\beta : \mathbb{P}(M_{A,E}) \xrightarrow{\sim} \mathbb{P}^m_S \end{array} \right\}_{/\ \mathrm{Iso.}}$$

which is obviously a PGL_{m+1}-torsor (principal bundle) over $\mathcal{A}^{\mathrm{alg}}_{g,d,N}$.
This reduces to show that $\mathcal{A}^{\mathrm{rig}}_{g,d,N}$ is representable because Mumford showed in his work on geometric invariant theory that $\mathcal{A}_{g,d,N}$ exists as a good geometric quotient of $\mathcal{A}^{\mathrm{rig}}_{g,d,N}$ modulo PGL_{m+1}. It will be quasi-projective if $\mathcal{A}^{\mathrm{rig}}_{g,d,N}$ is quasi-projective.
Now each quadruple (A, E, ξ, β) defines an embedding $\varphi : A \hookrightarrow \mathbb{P}^m_S$ of varieties over S. The image of φ has $2g + 1$ marked points namely the images of 0, and a the images of a basis of $(\mathbb{Z}/N\mathbb{Z})^{2g}$ under ξ. The

polarization can be reconstructed from the image since $\varphi^*\mathcal{O}(1) \cong s(E)^{\otimes 3}$ and $\Lambda(s(E)) = 2E$. This shows that we have an embedding of functors

$$\mathcal{A}^{\mathrm{rig}}_{g,d,N} \hookrightarrow \mathrm{Hilb}^{2g+1}(\mathbb{P}^m) \tag{20}$$

where

$$\mathrm{Hilb}^{2g+1}(\mathbb{P}^m)(S) = \left\{ \begin{array}{c} \text{subvarieties } V \text{ of } \mathbb{P}^m_S \\ +\ (2g+1) \text{ marked sections } S \to V \end{array} \right\}.$$

The functor $\mathrm{Hilb}^{2g+1}(\mathbb{P}^m)$ is not nice enough as stated here, but one can fix also the Hilbert polynomial

$$p : t \mapsto \chi(\mathcal{O}_V(t))$$

of the subvarieties parametrized. Indeed, by Riemann-Roch we have

$$\chi(\mathcal{O}_{\varphi(A)}(t)) = \chi(s(E)^{\otimes 3t}) = (6t)^g d$$

hence $p(t) = 6^g d \cdot t^g$. Furthermore one should work with schemes to get a nice representable functor:

$$\mathrm{Hilb}^{6^g d\cdot t^g, 2g+1}(\mathbb{P}^m)(S) = \left\{ \begin{array}{c} \text{subschemes } V \text{ of } \mathbb{P}^m_S \\ \text{having point-wise the Hilbert-polynomial } t \mapsto 6^g d t^g \\ +\ (2g+1) \text{ marked sections } S \to V \end{array} \right\}$$

This is indeed representable by a *projective* variety (of finite type) over $\mathbb{Q}$. This leaves the following steps

1. Show that the embedding (20) is open. This implies that $\mathcal{A}^{\mathrm{rig}}_{g,d,N}$ is representable as well and is a quasi-projective variety.
2. Show that $\mathcal{A}_{g,d,N}$ is smooth. This is the subject of deformation theory.

□

3.4 The Shimura variety associated with $(\mathrm{GSp}_{2g}, \mathbb{H}_g^{\pm})$

3.4.1 Adelic motivation

3.49. By construction, we have an action of $\mathrm{GSp}(\phi_d, \mathbb{Z}/N\mathbb{Z})$ from the right on the individual

$$\mathcal{A}_{g,d,N} \qquad \text{resp.} \quad \mathcal{A}^{\mathrm{alg}}_{g,d,N}$$

by pre-composing the level structure. This is also very much apparent using the analytic description

$$\mathcal{A}_{g,d,N} \cong \mathrm{GSp}(\phi_d, \mathbb{Z})\backslash \mathbb{H}_g^{\pm} \times \mathrm{GSp}(\phi_d, \mathbb{Z}/N\mathbb{Z}).$$

Here this action is just given by multiplying the second component from the right. However, the moduli description shows that the corresponding morphism of varieties is defined over $\mathbb{Q}$. Recall that the quotient above decomposes as a finite union

$$\mathcal{A}_{g,d,N} = \bigcup_i \Gamma(N)\backslash \mathbb{H}_g^{\pm}$$

where the $\Gamma(N)$ is the principal congruence subgroup of level N in $\mathrm{GSp}(\phi_d, \mathbb{Z})$. Considering all these quotients together, we even have an action of $\mathrm{GSp}(\phi_d, \mathbb{Q})$: The action of $\mathrm{GSp}(\phi_d, \mathbb{R})$ on $\mathbb{H}^{\pm}$ induces for each element $\gamma \in \mathrm{GSp}(\mathbb{Q})$ and for every congruence subgroup $\Gamma \subset \mathrm{GSp}(\mathbb{Q})$ a morphism

$$\begin{array}{rcl} \Gamma\backslash \mathbb{H}_g^{\pm} & \to & \gamma\Gamma\gamma^{-1}\backslash \mathbb{H}_g^{\pm} \\ \Gamma\tau & \mapsto & (\gamma\Gamma\gamma^{-1})(\gamma\tau) \end{array}$$

For example, this induces an action of $\mathrm{GSp}(\phi_d, \mathbb{Q})$ on the projective limit of all quotients $\Gamma\backslash \mathbb{H}_g^{\pm}$ where Γ runs through all congruence subgroups. This action boils down to the (inverse of the) previous action of $\mathrm{GSp}(\phi_d, \mathbb{Z}/N\mathbb{Z})$ if $\gamma \in \mathrm{GSp}(\phi_d, \mathbb{Z})$ and $\Gamma = \Gamma(N)$. However, what is the modular interpretation, if γ is more general? We have seen an example of this discussing Hecke operators in the introduction.

3.50. We will now review this example setting $g=1$ and ignoring the polarizations, which are canonical if $g=1$. Let p be a prime. We looked at the morphism

$$\Gamma_0(p)\backslash\mathbb{H}_1^{\pm} \to \mathrm{GL}_2(\mathbb{Z})\backslash\mathbb{H}_1^{\pm}$$

induced by the matrix $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$. We can compose the morphism above with the projection to get a morphism

$$\mathrm{GL}_2(\mathbb{Z})\backslash\mathbb{H}_1^{\pm} \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z})\backslash\mathbb{H}_1^{\pm}$$

i.e. a morphism

$$\mathcal{A}_{1,p} \to \mathcal{A}_{1,1}$$

(ignoring here the fact that $N \geq 3$ does not hold for the second space). Its modular description is the map

$$[E,\xi] \to [E/\mathrm{span}(\xi(\begin{pmatrix} 1 \\ 0 \end{pmatrix}))]. \tag{21}$$

This is seen as follows: The Riemann surface $\Gamma_0(p)\backslash\mathbb{H}_1^{\pm}$ also has a modular interpretation. It parametrizes elliptic curves together with the choice of a order-p-subgroup $\wp \subset E$. The modular interpretation of the projection is the map $(E,\xi) \mapsto (E, \mathrm{span}(\xi(\begin{pmatrix} 1 \\ 0 \end{pmatrix})))$. To see the effect of multiplication by $\begin{pmatrix} p & \\ & 1 \end{pmatrix}$ on $\Gamma_0(p)\backslash\mathbb{H}^{\pm}$ we have to recall how a moduli point $(E,\wp)$ is identified with a point in $\Gamma_0(p)\backslash\mathbb{H}^{\pm}$. For this we choose a trivialization of $\Lambda = H_1(E,\mathbb{Z})$ such that we have a diagram:

$$\begin{tikzcd} \Lambda \arrow[r, "\beta"] \arrow[d, "p"] & \mathbb{Z}^2 \arrow[d] \\ \frac{1}{p}\Lambda/\Lambda \arrow[r, "\overline{\beta}"] & (\mathbb{Z}/p\mathbb{Z})^2 \end{tikzcd}$$

which has the property $\overline{\beta}(\wp) = \mathrm{span}(\begin{pmatrix} 1 \\ 0 \end{pmatrix})$. Such a β is determined up to multiplication with a matrix in $\Gamma_0(p)$ and the complex structure on $\Lambda_{\mathbb{R}}$ which determines E is transported to a complex structure on $\mathbb{R}^2$ parametrized by a point in $\mathbb{H}^{\pm}$.
Hence E is isomorphic to

$$(\mathbb{R}^2,\tau)/\mathbb{Z}^2$$

where τ is a complex structure $\tau \in \mathbb{H}^{\pm}$ and the translated point corresponds to the elliptic curve

$$(\mathbb{R}^2, \begin{pmatrix} p & \\ & 1 \end{pmatrix}\tau)/\mathbb{Z}^2 \cong (\mathbb{R}^2,\tau)/\begin{pmatrix} p^{-1} & \\ & 1 \end{pmatrix}\mathbb{Z}^2$$

Observe that there is an isogeny (induced by the identity)

$$(\mathbb{R}^2,\tau)/\mathbb{Z}^2 \to (\mathbb{R}^2,\tau)/\begin{pmatrix} p^{-1} & \\ & 1 \end{pmatrix}\mathbb{Z}$$

whose kernel is precisely the subgroup $\mathrm{span}(\begin{pmatrix} 1 \\ 0 \end{pmatrix})$ corresponding to $\wp$ under the isomorphism β.
The map (21)

1. changes the isomorphism class of E (as opposed to the action of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ considered before), however within the same isogeny class.
2. uses of the level structure only the information of the order p subgroup $\mathrm{span}(\xi(\begin{pmatrix} 1 \\ 0 \end{pmatrix}))$.

2. is not a surprise, since the modular description $\Gamma_0(p)\backslash\mathbb{H}_1^{\pm}$ is "elliptic curves + order-p-subgroup" or equivalently "isogenies of elliptic curves of order p".
We will now explain how the adelic language allows to unify the actions of $\mathrm{GL}_2(\mathbb{Q})$ and the $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ in a very elegant way. The story starts by replacing the lattice considered above by the Tate module: The exact sequence

$$0 \longrightarrow \wp \longrightarrow A \longrightarrow A/\wp \longrightarrow 0$$

induces the sequence of Tate modules (cf. Corollary 3.43, 4.)

$$0 \longrightarrow T(E) \longrightarrow T(E/\wp) \longrightarrow \wp \longrightarrow 0. \tag{22}$$

The level structure can be lifted to an isomorphism

$$\begin{tikzcd} T(A) \arrow[r, "\widetilde{\xi}"] \arrow[d] & \widehat{\mathbb{Z}}^2 \arrow[d] \\ A[p] \arrow[r, "\xi"] & (\mathbb{Z}/p\mathbb{Z})^2 \end{tikzcd}$$

which is uniquely determined up to multiplying matrices in $K(N) \subset \mathrm{GL}_2(\widehat{\mathbb{Z}})$.
Using $\widetilde{\xi}$ we may translate the sequence (22) to the sequence:

$$0 \longrightarrow \widehat{\mathbb{Z}}^2 \longrightarrow \begin{pmatrix} p^{-1} & \\ & 1 \end{pmatrix} \widehat{\mathbb{Z}}^2 \longrightarrow \mathrm{span}(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) \longrightarrow 0$$

In other words: the quotient $A/\wp$ can be described by changing the Tate-module $T(E)$ identified with $\widehat{\mathbb{Z}}^2$ to the superlattice in $T(E) \otimes \mathbb{Q}$ identified with $\begin{pmatrix} p^{-1} & \\ & 1 \end{pmatrix} \widehat{\mathbb{Z}}^2 \subset \widehat{\mathbb{Z}}^2 \otimes \mathbb{Q}$.

3.51. To make this more clear, one should work with elliptic curves and, more generally, Abelian varieties up to isogeny, i.e. considering isogenous Abelian varieties as isomorphic. This can be done by defining a category

$$\mathbb{Q} \otimes [\ \textbf{Abelian varieties} \ / \ k\ ]$$

where the objects are Abelian varieties A and the morphism sets are $\mathrm{Hom}(A, B) \otimes \mathbb{Q}$.

In this category an isogeny becomes an isomorphism because we have the following

Lemma 3.52. *For each isogeny $\phi : A \to B$ defined over k there is an isogeny $\phi' : B \to A$ defined over k such that $\phi'\phi = [N]$.*

Proof. Since $\ker(\phi)$ is a finite subgroup (scheme) of A by definition of isogeny there is an N such that $N \ker(\phi) = 0$[11] and a diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker(\phi) \arrow[r] \arrow[d, hook] & A \arrow[r, "\phi"] \arrow[d, equal] & B \arrow[r] \arrow[d, dotted] & 0 \\ 0 \arrow[r] & A[N] \arrow[r] & A \arrow[r, "{[N]}"] & A \arrow[r] & 0 \end{tikzcd}$$

The dotted arrow is then constructed using the universal property of quotients. □

In our isogeny category we have hence $(\phi' \otimes \frac{1}{N})(\phi \otimes 1) = \mathrm{id}_A$. Applying the Lemma to ϕ' we see that also $(\phi \otimes 1)(\phi' \otimes \frac{1}{N}) = \mathrm{id}_B$. Any isomorphism in the isogeny category will be called a **$\mathbb{Q}$-isogeny**.
Now observe that the functor

$$V : A \mapsto T(A) \otimes \mathbb{Q}$$

is well-defined as a functor on the isogeny category $\mathbb{Q} \otimes [\ \textbf{Abelian varieties} \ / \ k\ ]$. The actual Tate module is always a $\widehat{\mathbb{Z}}$-lattice in $T(A) \subset V(A)$. A may be reconstructed from it:

[11]If we work over an arbitrary field and with group schemes, then this assertion is not completely trivial!

Proposition 3.53. *Let k be a field of char 0. There is an equivalence of categories:*

$$\left[\begin{array}{c}\textbf{\textit{Abelian varieties }} A/k \textbf{\textit{ up to }} \mathbb{Q}\textbf{\textit{-isogeny}} \\ +\ \mathrm{Gal}(\overline{k}/k)\textbf{\textit{-stable lattice }} L \subset V(A_{\overline{k}})\end{array}\right] \cong [\ \textbf{\textit{Abelian varieties / }} k\ ]$$

In the categories on the left hand side morphisms have to map the lattice L to the corresponding lattice L'.

Proof. We concentrate one the case $k = \overline{k}$ first. The functor (going from right to left) is induced by the identity $A \mapsto A$ where $V(A)$ is equipped with the lattice given by the Tate module $T(A)$. It is obviously faithful because $\mathrm{Hom}(A, B)$ is torsion-free. It is full, for consider a map $\phi \otimes q$ with $q \in \mathbb{Q}$ and $\phi : A \to B$ a $\mathbb{Q}$-isogeny which maps $T(A)$ to $T(B)$. W.l.o.g. we may assume that $q = \frac{1}{N}$. Therefore it has the property $\phi(T(A)) \subset NT(B)$. From this it follows that $\phi(A[N]) = 0$ and hence ϕ factors through the map $[N]$ by an argument similar to the one of Lemma 3.52.
We now show that the functor is essentially surjective. By multiplying L with a rational number, we may assume w.l.o.g. that L contains $T(A)$ (see the Lemma below for a better understanding of $\widehat{\mathbb{Z}}$-lattices).
Choose an N such that N kills $K := T(A)/L$ We therefore get a sequence

$$\begin{tikzcd} 0 \arrow[r] & T(A) \arrow[r, hook] \arrow[d, equal] & L \arrow[r] \arrow[d, hook] & K = T(A)/L \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & T(A) \arrow[r, hook] & \frac{1}{N}T(A) \arrow[r] & A[N] \arrow[r] & 0 \end{tikzcd}$$

Because of the embedding into $A[N]$, we can form the quotient $A' = A/K$ and get a canonical identification

$$T(A) \cong L.$$

In other words $\phi : A \to A'$ induces an isomorphism $(A, L) \cong (A', T(A'))$. Therefore the functor is essentially surjective.
In the case that k is not algebraically closed, observe that if the lattice L is Galois stable, the subgroup K is Galois stable (as a subgroup) and hence defines a subgroup variety *defined over k*. The rest of the construction then works the same way. $\square$

Instead of considering elliptic curves and level-N-structures we can therefore consider an elliptic curves up to $\mathbb{Q}$-isogeny and an isomorphism

$$\widetilde{\xi} : \widehat{\mathbb{Z}}^2 \otimes \mathbb{Q} = \mathbb{A}_f^2 \to V(E)$$

modulo $K(N)$. This reconstructs an actual elliptic curve E' in the $\mathbb{Q}$-isogeny class of E which has the property that $(E', T(E'))$ is isomorphic to $(E, \widetilde{\xi}(\widehat{\mathbb{Z}}^2))$. The level-$N$-structure is the reconstructed by the diagram

$$\begin{tikzcd} T(E') \arrow[r, "\widetilde{\xi}"] \arrow[d] & \widehat{\mathbb{Z}}^2 \arrow[d] \\ E'[p] \arrow[r, "\xi"] & (\mathbb{Z}/p\mathbb{Z})^2 \end{tikzcd}$$

Considering the level structure

$$\widetilde{\xi} : \widehat{\mathbb{Z}}^2 \otimes \mathbb{Q} = \mathbb{A}_f^2 \to V(E)$$

now has the advantage that we have an (right) action of $\mathrm{GL}_2(\mathbb{A}_f)$ on it, which combines the action of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ (via a lift to $\mathrm{GL}_2(\widehat{\mathbb{Z}})$) and the action of $\mathrm{GL}_2(\mathbb{Q})$!
Above we worked with $\widehat{\mathbb{Z}}$-lattices in $\mathbb{A}_f^{2g}$ as if they were actually $\mathbb{Z}$-lattices in $\mathbb{Q}^{2g}$. This is legitimate because of the following statement

Lemma 3.54. *There is a bijection*

$$\begin{array}{rcl} \{\mathbb{Z}\textit{-lattices in } \mathbb{Q}^n\} & \cong & \{\widehat{\mathbb{Z}}\textit{-lattices in } \mathbb{A}_f^n\} \\ L & \mapsto & L \otimes \widehat{\mathbb{Z}} \\ \mathcal{L} \cap \mathbb{Q}^n & \leftarrow\!\shortmid & \mathcal{L} \end{array}$$

Proof. The statement is equivalent to

$$\mathrm{GL}_n(\mathbb{Q})/\mathrm{GL}_n(\mathbb{Z}) \cong \mathrm{GL}_n(\mathbb{A}_f)/\mathrm{GL}_n(\widehat{\mathbb{Z}})$$

because the group $\mathrm{GL}_n(\mathbb{Q})$ (resp. $\mathrm{GL}_2(\mathbb{A}_f)$) acts transitively on the set of $\mathbb{Z}$-lattices (resp. $\widehat{\mathbb{Z}}$-lattices). This in turn is equivalent to

$$\mathrm{GL}_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{A}_f)/\mathrm{GL}_n(\widehat{\mathbb{Z}}) = \{1\}.$$

This is saying that **the class number of GL_n is 1**. It follows immediately from the statements

$$\mathbb{G}_m(\mathbb{Q})\backslash\mathbb{G}_m(\mathbb{A}_f)/\mathbb{G}_m(\widehat{\mathbb{Z}}) = \{1\}, \qquad \mathrm{SL}_n(\mathbb{Q})\backslash \mathrm{SL}_n(\mathbb{A}_f)/\mathrm{SL}_n(\widehat{\mathbb{Z}}) = \{1\}.$$

The first just translates to the class number of $\mathbb{Q}$ being 1. The second follows from the stronger statement (denoted by **strong approximation**) that

$$\mathrm{SL}_n(\mathbb{Q}) \hookrightarrow \mathrm{SL}_n(\mathbb{A}_f)$$

is dense. For SL_n it follows from the fact that this algebraic group is generated by its 1-paramenter subgroups

$$\left\{\begin{pmatrix} 1 & & & \\ & \ddots & x & \\ & & \ddots & \\ & & & 1 \end{pmatrix}\right\} \cong \mathbb{G}_a$$

and that $\mathbb{Q} = \mathbb{G}_a(\mathbb{Q}) \hookrightarrow \mathbb{A}_f = \mathbb{G}_a(\mathbb{A}_f)$ is dense. The latter assertion is basically equivalent to the Chinese remainder theorem. ☐

3.4.2 The adelic moduli problems

3.55. The adelic formulation has also the advantage that the distinction into different types d and the level structures become intertwined. We choose the standard-form ϕ_1 on our reference vector space $\mathbb{Q}^{2g}$ and denote GSp the corresponding algebraic group.
We now define an adelic version of the algebraic and analytic moduli spaces $\mathcal{A}_{g,d,N}$. Consider any compact open subgroup $K \subset \mathrm{GSp}(\mathbb{A}_f)$. Define[12]

$$\mathcal{A}^{\mathrm{alg}}_{g,K}(S) \cong \left\{ \begin{array}{c} A \text{ family of Abelian varieties over } S \text{ of dimension } g \text{ up to } \mathbb{Q}\text{-isogeny} \\ E \in \mathrm{NS}(A)\otimes\mathbb{Q} \text{ a polarization} \\ \xi : (\mathbb{A}_f)^{2g} \xrightarrow{\sim} V(A_{\overline{s}}) \text{ class of symplectic similitudes modulo } K \text{ defined over } S \end{array} \right\}_{/\text{ Iso.}}$$

Here $NS(A)$ for an Abelian variety A up to $\mathbb{Q}$-isogeny is defined as $\mathrm{Hom}(A, A^\vee)\otimes\mathbb{Q}$ and "polarization" means that there is an actual polarization $\widetilde{E}$ of A such that $E = \widetilde{E}\otimes q$ for some $q \in \mathbb{Q}$. The property "symplectic similitude" is meant to hold w.r.t. the standard form ϕ_1 on $(\mathbb{A}_f)^{2g}$ and the Weil pairing on $(\mathbb{A}_f)^{2g}$. The isomorphisms (which are now given by $\mathbb{Q}$-isogenies) have to respect the respective polarizations E up to scalar in $\mathbb{Q}^*$.
Similarly we define analytically (without going into technical details):

$$\mathcal{A}_{g,K}(S) \cong \left\{ \begin{array}{c} \pi : X \to S \text{ family of complex tori of dim } g \text{ up to } \mathbb{Q}\text{-isogeny} \\ E \in R^2\pi_*\mathbb{Q}(1) \text{ which is fibrewise a polarization} \\ \xi : (\mathbb{A}_f)^{2g}_S \xrightarrow{\sim} V(X) \text{ class of symplectic similtude of } \mathbb{A}_f\text{-local systems mod } K \text{ over } S \end{array} \right\}_{/\text{ Iso.}}$$

The same procedure as in section 3.2.4 shows that, as a space, we have

$$\mathcal{A}_{g,K} \cong \mathrm{GSp}_{2g}(\mathbb{Q})\backslash\mathbb{H}_g^{\pm} \times (\mathrm{GSp}_{2g}(\mathbb{A}_f)/K).$$

[12] Here $V(A)$ has to be considered over a generic geometric point $\overline{s}$ of S (for each connected component) and 'defined over S' means that the class modulo K has to be invariant under the corresponding etale fundamental group $\pi_1(S, \overline{s})$.

On the moduli description and this explicit description the right action of $\mathrm{GSp}(\mathbb{A}_f)$ becomes apparent. We will now show that we have $\mathcal{A}_{g,K_d(N)} \cong \mathcal{A}_{g,d,N}$ for the group

$$K_d(N) = \{X \in \mathrm{GSp}(\widehat{L}_d) \mid g \equiv 1(N)\}.$$

Here instead of varying the form to ϕ_d on $\mathbb{Z}^{2g}$ we from now on fix ϕ_1 and vary the lattice from $\mathbb{Z}^{2g}$ to

$$L_d := \begin{pmatrix} D & \\ & 1 \end{pmatrix} \mathbb{Z}^{2g}.$$

This is of type d. We denote $\widehat{L}_d := L_d \otimes \widehat{\mathbb{Z}}$.
Furthermore these subgroups $K_d(N)$ (for varying d, N or even fixing $d = (1, \dots, 1)$ or two relatively coprime tupels d_1, d_2) form a cofinal system, hence the projective limit of the $\mathcal{A}_{g,K}$ is equal to the projective limit of the $\mathcal{A}_{g,1,N}$ resp. to the projective limit of the $\mathcal{A}_{g,d_1,N}$ and $\mathcal{A}_{g,d_2,N}$.

3.56. The equality of the classical moduli problem and the adelic moduli problems for $K = K_d(N)$ are summarized by the following commutative diagram:

$$\begin{tikzcd}
\mathcal{A}^{\mathrm{alg}}_{g,d,N}(\mathbb{C}) \arrow[r, "\sim"] \arrow[d, "\sim"', "1"] & \mathcal{A}_{g,d,N} \arrow[r, "\sim"] \arrow[d, "\sim"', "2"] & \mathrm{GSp}(L_d)\backslash \mathbb{H}_g^{\pm} \times \mathrm{GSp}(L_d \otimes \mathbb{Z}/N\mathbb{Z}) \arrow[d, "\sim"', "3"] \\
\mathcal{A}^{\mathrm{alg}}_{g,K_d(N)}(\mathbb{C}) \arrow[r, "\sim"] & \mathcal{A}_{g,K_d(N)} \arrow[r, "\sim"] & \mathrm{GSp}_{2g}(\mathbb{Q})\backslash \mathbb{H}_g^{\pm} \times (\mathrm{GSp}_{2g}(\mathbb{A}_f)/K_d(N))
\end{tikzcd}$$

The translations 1 and 2 are similar. Therefore we will only describe the algebraic version 1 and its inverse:
Let (A, E, ξ) be a moduli-point in $\mathcal{A}^{\mathrm{alg}}_{g,d,N}(\mathbb{C})$. The level-structure ξ lifts to a level structure $\widetilde{\xi}$:

$$\begin{tikzcd}
\widehat{L}_d \arrow[r, "\widetilde{\xi}"] \arrow[d] & T(A) \arrow[d] \\
L_d \otimes \mathbb{Z}/N\mathbb{Z} \arrow[r, "\xi"] & A[N]
\end{tikzcd}$$

This follows because the map $\mathrm{GSp}(\widehat{L}_d) \to \mathrm{GSp}(L_d \otimes \mathbb{Z}/N\mathbb{Z})$ is surjective. Tensoring with $\mathbb{Q}$ we hence get a level-structure

$$\mathbb{A}_f^2 \xrightarrow{\widetilde{\xi}\otimes\mathbb{Q}} V(A).$$

The tripel $(A, E\otimes\mathbb{Q}, \widetilde{\xi}\otimes\mathbb{Q})$ defines a moduli-point in $\mathcal{A}^{\mathrm{alg}}_{g,K_d(N)}(\mathbb{C})$ which is well-defined because $\widetilde{\xi}$ is obviously well-defined up to multiplication with element from $K_d(N)$.
Let now (A, E, η) be a moduli-point in $\mathcal{A}^{\mathrm{alg}}_{g,K_d(N)}(\mathbb{C})$. By Proposition 3.53 there is a $\mathbb{Q}$-isogeny $\rho : \mathcal{A} \to \mathcal{A}'$ such that the composition

$$\widetilde{\xi} : \mathbb{A}_f^{2g} \xrightarrow{\eta} V(A) \xrightarrow{V(\rho)} V(A')$$

has the property that $\widetilde{\xi}(\widehat{L}_d) = T(A')$ (for η this does not need to hold). Define E' the be the $\mathbb{Q}$-isogeny composition of

$$A' \xrightarrow{\rho^{-1}} A \xrightarrow{E} A^\vee \xrightarrow{(\rho^{-1})^\vee} (A')^\vee .$$

The moduli-point (A, E, η) is by construction equal to $(A', E', \widetilde{\xi})$. We claim that there is a $q \in \mathbb{Q}$ such that $qE' = E'' \otimes 1$ for a polarization E'' of type d on A'. Indeed, $V(E')$ is an isomorphism

$$V(A') \to V((A')^\vee) \cong V(A')^*(1)$$

which induces on $V(A')$ a λ-multiple (for some $\lambda \in \mathbb{A}_f^*$) of the form ϕ_d transported to $V(A')$ via $\widetilde{\xi}$. Write $\lambda = qz$ with $q \in \mathbb{Q}^*$ and $z \in \widehat{\mathbb{Z}}^*$. The $\mathbb{Q}$-isogeny qE' has then the property of mapping $T(A')$ to $T((A')^\vee)$. By Proposition 3.53 it is therefore of the form $E'' \otimes 1$ for an actual morphism $A' \to (A')^\vee$. E'' induces on $T(A')$ a symplectic form of type d hence by definition is a polarization of type d.

Finally, the moduli-point (A', E'', ξ), in which ξ is the reduction of $\widetilde{\xi}$ modulo N, is in $\mathcal{A}_{g,d,N}^{\mathrm{alg}}(\mathbb{C})$. One checks that these maps are indeed inverse to each other. A refinement of the construction allows to do the same in families to get an isomorphism of moduli functors:

$$\mathcal{A}_{g,K_d(N)}^{\mathrm{alg}} \cong \mathcal{A}_{g,d,N}^{\mathrm{alg}}.$$

This shows that also the left hand side functor is representable. From this one can infer the representability of all $\mathcal{A}_{g,K}^{\mathrm{alg}}$, *provided that K is small enough*, as follows: Because the $K_1(N)$'s form a cofinal system of the compact open subgroups, there is an N such that $K_1(N) \subset K$ and we can form the quotient[13]

$$\mathcal{A}_{g,K_1(N)}^{\mathrm{alg}}/(K_1(N)\backslash K)$$

(even if $K_1(N)$ is not normal in K one can make sense of this as the quotient modulo an equivalence relation) and this will represent the moduli functor $\mathcal{A}_{g,K}^{\mathrm{alg}}$.

The map 1 (or equivalently 2) can be described directly on the corresponding analytic moduli space. This is the map 3 and is given as follows:

Let a point $[\tau, \xi] \in \mathrm{GSp}(L_d)\backslash \mathbb{H}_g^\pm \times \mathrm{GSp}(L_d \otimes \mathbb{Z}/N\mathbb{Z})$ be given. ξ lifts as before to an element $\widetilde{\xi} \in \mathrm{GSp}(\widehat{L}_d)$. Hence we get a point $[\tau, \widetilde{\xi}] \in \mathrm{GSp}_{2g}(\mathbb{Q})\backslash \mathbb{H}_g^\pm \times (\mathrm{GSp}_{2g}(\mathbb{A}_f)/K_d(N))$.

Conversely let a point $[\tau, \eta] \in \mathrm{GSp}_{2g}(\mathbb{Q})\backslash \mathbb{H}_g^\pm \times (\mathrm{GSp}_{2g}(\mathbb{A}_f)/K_d(N)$ be given. By Lemma 3.57 below we can write $\eta = \eta_{\mathbb{Q}} \cdot \widetilde{\xi}$ for $\widetilde{\xi} \in K_d(1) = \mathrm{GSp}(\widehat{L}_d)$. Therefore we have

$$[\tau, \eta] = [\tau, \eta_{\mathbb{Q}} \cdot \widetilde{\xi}] = [\eta_{\mathbb{Q}}^{-1} \cdot \tau, \widetilde{\xi}]$$

which we map to the moduli point $[\eta_{\mathbb{Q}}^{-1} \cdot \tau, \xi] \in \mathrm{GSp}(L_d)\backslash \mathbb{H}_g^\pm \times \mathrm{GSp}(L_d \otimes \mathbb{Z}/N\mathbb{Z})$ in which ξ is the reduction of $\widetilde{\xi}$ modulo N.

Lemma 3.57. *We have*

$$\mathrm{GSp}_{2g}(\mathbb{Q})\backslash \mathrm{GSp}_{2g}(\mathbb{A}_f)/K_d(1) = \{1\}$$

i.e. the class number of the group GSp_{2g} *w.r.t. the lattice* L_d *is one.*

Proof. This can be proven the same way as Lemma 3.54 using this time strong approximation for Sp_{2g} instead of SL_{2g}. However, it follows more easily from the fact that both

$$\mathrm{GSp}(\mathbb{Q})L_d$$

and

$$\mathrm{GSp}(\mathbb{A}_f)\widehat{L}_d$$

are equal (using the bijection of Lemma 3.54) to the set of lattices of type d. Note that "of type d" can in both cases be expressed by saying

$$L_d^*/L_d \quad (= \widehat{L}_d^*/\widehat{L}_d) \quad \cong (\mathbb{Z}/d_1\mathbb{Z})^2 \times \cdots \times (\mathbb{Z}/d_n\mathbb{Z})^2.$$

∎

[13] of etale sheaves, to be precise

3.4.3 A model of the Shimura variety

In the last section we constructed a smooth quasi-projective variety $\mathcal{A}^{\mathrm{alg}}_{g,K}$, defined over $\mathbb{Q}$ for each small enough compact open subgroup K which is a model of the analytic space

$$\mathrm{GSp}_{2g}(\mathbb{Q})\backslash\mathbb{H}_g^{\pm} \times (\mathrm{GSp}_{2g}(\mathbb{A}_f)/K).$$

Furthermore the adelic group $\mathrm{GSp}_{2g}(\mathbb{A}_f)$ acts on this system by morphisms of algebraic varieties defined over $\mathbb{Q}$. Deligne gave the following abstraction of this situation:

Lemma 3.58. *Let G be a totally disconnected locally compact group. It is equivalent to give*

1. *A scheme $M_{\mathbb{Q}}$ (not of finite type!) over $\mathrm{spec}(\mathbb{Q})$ with continuous*[14] *right G-action such that the quotient $M_{\mathbb{Q}}/K$ exists for all sufficiently small compact open subgroups $K \subset G$ and is a smooth quasi-projective variety.*
2. *For each sufficiently small compact open subgroup $K \subset G$ a smooth quasi-projective variety M_K and for each K, L compact open subgroups, and for each $g \in G$ with $g^{-1}Kg \subset L$ a morphism of $\mathbb{Q}$-varieties*

$$J_{K,L}(g) : M_K \to M_L$$

such that

   (a) $J_{L,M}(h) \circ J_{K,L}(g) = J_{K,M}(hg)$,

   (b) $J_{K,K}(g) = \mathrm{id}_{M_K}$ *if* $g \in K$,

   (c) For each normal subgroup $K \lhd L$ the morphisms $J_{K,K}$ define an action of $K\backslash L$ on M_K such that the induced map

$$J_{K,L}(1) : M_K/(K\backslash L) \to M_L$$

   is an isomorphism.

3. *For each K in a cofinal system the same data as in 2.*

Definition 3.59. *A system of smooth quasi-projective $\mathbb{Q}$-varieties $\{M_K\}_K$ with morphisms $J_{K,L}(g)$ as in the previous lemma (for $G = \mathrm{GSp}_{2g}(\mathbb{A}_f)$) together with isomorphisms*

$$M_K(\mathbb{C}) \cong \mathrm{GSp}_{2g}(\mathbb{Q})\backslash\mathbb{H}_g^{\pm} \times (\mathrm{GSp}_{2g}(\mathbb{A}_f)/K)$$

for every sufficiently small K is called a **model** *of the Shimura variety $\{\mathrm{GSp}_{2g}(\mathbb{Q})\backslash\mathbb{H}_g^{\pm} \times (\mathrm{GSp}_{2g}(\mathbb{A}_f)/K)\}_K$ if the isomorphisms are compatible with the $\mathrm{GSp}_{2g}(\mathbb{A}_f)$-action, i.e. if for all K, L, g as in 2. of the lemma the following diagram is commutative:*

$$\begin{tikzcd}
M_K(\mathbb{C}) \arrow[r, "\sim"] \arrow[d, "J_{K,L}(g)(\mathbb{C})"'] & \mathrm{GSp}_{2g}(\mathbb{Q})\backslash\mathbb{H}_g^{\pm} \times (\mathrm{GSp}_{2g}(\mathbb{A}_f)/K) \arrow[d, "{[\tau,h]\mapsto[\tau,h\cdot g]}"] \\
M_L(\mathbb{C}) \arrow[r, "\sim"] & \mathrm{GSp}_{2g}(\mathbb{Q})\backslash\mathbb{H}_g^{\pm} \times (\mathrm{GSp}_{2g}(\mathbb{A}_f)/L)
\end{tikzcd}$$

We will later generalize this Definition to more general Shimura varieties.
According to the Lemma one could also work with the projective limits of both objects instead. For the Shimura variety we just have

$$\lim_K \mathrm{GSp}_{2g}(\mathbb{Q})\backslash\mathbb{H}_g^{\pm} \times (\mathrm{GSp}_{2g}(\mathbb{A}_f)/K) = \mathrm{GSp}_{2g}(\mathbb{Q})\backslash\mathbb{H}_g^{\pm} \times \mathrm{GSp}_{2g}(\mathbb{A}_f).$$

We will not do this, however.
We may summarize the discussion of the previous sections by

[14] This means that $M_{\mathbb{Q}} \cong \lim_K M_{\mathbb{Q}}/K$.

Corollary 3.60. *The collection*

$$\{\mathcal{A}_{g,K}\}_K$$

(together with all the morphisms $J_{K,L}(g)\dots$) for all sufficiently small compact open subgroups $K \subset \mathrm{GSp}(\mathbb{A}_f)$ form a model of the Shimura variety

$$\{\mathrm{GSp}_{2g}(\mathbb{Q})\backslash\mathbb{H}_g^{\pm} \times (\mathrm{GSp}_{2g}(\mathbb{A}_f)/K)\}_K$$

in the sense of Definition 3.59.

3.4.4 $\{\mathcal{A}_{g,K}\}_K$ is a canonical model

One of the goals of the theory of Shimura and Deligne was to characterize the model

$$\{\mathcal{A}_{g,K}\}_K$$

of the Shimura variety

$$\{\mathrm{GSp}_{2g}(\mathbb{Q})\backslash\mathbb{H}_g^{\pm} \times (\mathrm{GSp}_{2g}(\mathbb{A}_f)/K)\}_K$$

purely in group theoretical terms because similar moduli descriptions that we used for this particular Shimura variety are (still) not available for all Shimura varieties. As mentioned in the introduction the key to achieve this it the theory of *complex multiplication*.

Definition 3.61. *Let F be a number field of degree $2g$ over $\mathbb{Q}$. Let $\mathcal{O} \subset F$ be an order. An Abelian variety A of dimension g defined over k has* **complex multiplication** *by $\mathcal{O}$ if there is an embedding*

$$\mathcal{O} \hookrightarrow \mathrm{End}(A).$$

We also say that A (considered up to $\mathbb{Q}$-isogeny) has complex multiplication by F if there is an embedding

$$F \hookrightarrow \mathrm{End}(A) \otimes \mathbb{Q}.$$

This implies that A has complex multiplication by some *order in F.*

The analogous Definition holds for complex tori.
Assume now $k = \mathbb{C}$. We have seen in the beginning of section **??** that endomorphisms of A or, equivalently, of the complex torus $X = A(\mathbb{C})$ are given by endomorphisms of the corresponding Hodge structure (i.e. complex structure). Like in Lemma 3.17 we may express this again in a representation theoretic way by saying that *the Hodge structure is compatible with the representation of a group*:
Choose a basis $F \cong \mathbb{Q}^{2g}$. This defines an embedding $\iota : F^* \hookrightarrow \mathrm{GL}_{2g}(\mathbb{Q})$ and, more generally, an embedding of algebraic groups

$$\iota : F^x := \mathrm{Res}^F_{\mathbb{Q}}\mathbb{G}_m \hookrightarrow \mathrm{GL}_{2g,\mathbb{Q}}\,.$$

It also induces a decomposition

$$\mathbb{C}^{2g} = \sum_{r \in \mathrm{Hom}(F,\mathbb{C})} \mathbb{C}^{(r)},$$

where F acts on $\mathbb{C}^{(r)}$ via the embedding r.

Lemma 3.62. *For a Hodge structure of weight -1 given by $\{V^{-1,0}, V^{0,-1}\}$ on $\mathbb{R}^{2g}$ with corresponding representation $h : \mathbb{S} \to \mathrm{GL}_{2g,\mathbb{R}}$ or corresponding complex torus $X = (\mathbb{R}^{2g}, J)/\mathbb{Z}^{2g}$ the following are equivalent*

1. *For all $x \in F$*

$$x \in \mathrm{End}(\mathbb{Q})^{0,0}$$

   i.e. x induces an endomorphism of Hodge structures (or complex structures).

2. $$V^{-1,0} = \sum_{r \in S} \mathbb{C}^{(r)}$$

   for a disjoint decomposition $\mathrm{Hom}(F, \mathbb{C}) = S \cup \overline{S}$.

3. *h factors through $\iota(F^x)$.*

4. *ι induces an embedding $F \hookrightarrow \mathrm{End}(X) \otimes \mathbb{Q}$, i.e. X has complex multiplication by F.*

We leave the proof as an exercise. The set S of embeddings that occurs in 2. is called the **CM-type** of the pair $X, \iota : F \hookrightarrow \mathrm{End}(X) \otimes \mathbb{Q}$. Even if A is an algebraic variety defined over $\mathbb{C}$ it can be described purely algebraically: It is uniquely determined by the existence of an F-equivariant isomorphism

$$\mathrm{Lie}(A) \cong \sum_{r \in S} \mathbb{C}^{(r)}$$

where F acts on $\mathrm{Lie}(A)$ via the induced action (Lie is a functor on Abelian varieties). Remember: In the analytic case, we have $V^{-1,0} = \mathrm{Lie}(X) = \mathrm{Lie}(A)$. There is also a purely algebraic proof of the fact that there is a disjoint union $\mathrm{Hom}(F, \mathbb{C}) = S \cup \overline{S}$. Analytically it follows already from the defining condition of Hodge structures: $V^{0,-1} = \overline{V^{-1,0}}$. S is also encoded in the homomorphism h (cf. 3.) via the bijection

$$\{S \mid \mathrm{Hom}(F, \mathbb{C}) = S \cup \overline{S}\} \cong \{h \in \mathrm{Hom}(\mathbb{S}, F^x_{\mathbb{R}}) \text{ of type } (-1,0),(0,-1)\}.$$

Proof. Note that $\mathrm{Hom}(\mathbb{S}, F^x_{\mathbb{R}}) = \mathrm{Hom}(X^*(F^x), X^*(\mathbb{S}))^{\mathrm{Gal}(\mathbb{C}|\mathbb{R})}$ and $X^*(F^x_{\mathbb{C}})$ is canonically identified with $\mathbb{Z}[\mathrm{Hom}(F, \mathbb{C})]$ and we have $X^*(\mathbb{S}_{\mathbb{C}}) \cong \mathbb{Z}^2$ corresponding to the weights. A morphism h is of type $(-1,0),(0,-1)$ if and only if the basis elements in $\mathrm{Hom}(F, \mathbb{C})$ are mapped to either $(-1,0)$ or $(0,-1)$ under $X^*(h)$. $\square$

3.63. Lemma 3.62 completely ignored the question whether X is actually algebraic, i.e. is the complex analytic manifold associated with an Abelian variety, or equivalently, whether X is polarizable. We have seen in Lemma 3.17 that this is governed mainly by a compatibility of the complex structure with a *symplectic structure*. Therefore we have to understand the interplay between these two compatibilities. This is a bit technical because of the following problem. Imagine that $X = E^g$ where E is an elliptic curve with $\mathrm{End}(E) = \mathcal{O}$ where $\mathcal{O}$ is an order in an imaginary quadratic field K. Then obviously $\mathrm{End}(A) \otimes \mathbb{Q} \cong \mathrm{Mat}_{g \times g}(K)$ which trivially contains plenty of fields F of degree $2g$. This is a pathological and arithmetically non-interesting situation (one gets nothing new apart from E). We have to investigate when precisely this happens:

Definition 3.64. *Let $S \subset \mathrm{Hom}(F, \mathbb{C})$ be a CM-type. We define a field K by*

$$\mathrm{Gal}(\overline{\mathbb{Q}}/K) = \{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \mid \widetilde{S} \circ \sigma = \widetilde{S}\}.$$

Here $\widetilde{S}$ is the set of all extensions of the embeddings in S to $\overline{\mathbb{Q}}$.

Lemma 3.65. *Assume that the morphism $h : \mathbb{S} \hookrightarrow F^x_{\mathbb{R}} \hookrightarrow \mathrm{GL}_{2g,\mathbb{R}}$ factors also through $\mathrm{GSp}_{2g,\mathbb{R}}$ in such a way that the associated complex torus (up to isogeny) $X \otimes \mathbb{Q}$ becomes polarized w.r.t. ϕ_1. Then we have:*

1. *The smallest subtorus of the form $(F')^x$ such that h factors through it is given by K^x:*

$$h : \mathbb{S} \hookrightarrow K^x_{\mathbb{R}} \hookrightarrow F^x_{\mathbb{R}}.$$

2. *$X \cong Y^d$, where $d = [F : K]$ and Y is* simple[15].

3. $\mathrm{End}(X) \otimes \mathbb{Q} = M_{d \times d}(K)$.

This gives us a simple combinatorial criterion (namely: $K = F$) for a complex torus or an Abelian variety with CM to be simple. In this case it follows from 3. that

$$\mathrm{GL}_{2g}(\mathbb{Q}) \cap \mathrm{Stab}(h, \mathrm{GL}_{2g}(\mathbb{R})) = F^*.$$

Since h factors through $\mathrm{GSp}_{2g,\mathbb{R}}$ both groups are invariant under taking adjoints w.r.t. ϕ_1. Therefore also F^* is invariant under taking adjoints and since F^* is Zariski dense in F^x we have

$${}^t(F^x) = F^x$$

where ^{t} denotes the adjoint w.r.t. ϕ_1. It is convenient to assume just this stability under taking adjoints instead of the simplicity of X. It implies

[15] i.e. does not contain any non-trivial complex subtorus

Lemma 3.66. *1. F is a CM-field, i.e. there exists an automorphism $\tau \in \mathrm{Aut}(F/\mathbb{Q})$ which in every complex embedding induces complex conjugation*[16]*.*

2. Under the embedding $\mathrm{Aut}(F/\mathbb{Q}) \hookrightarrow \mathrm{Aut}(F^x)$, the automorphism τ corresponds to the adjunction w.r.t. the symplectic form and using the isomorphism $F \cong \mathbb{Q}^{2g}$ (which defined the embedding $F^x \hookrightarrow \mathrm{GL}_{2g,\mathbb{Q}}$), we have

$$\phi_1(x,y) = \mathrm{tr}(x\delta({}^\tau y))$$

with some $\delta \in F^*$ with ${}^\tau\delta = -\delta$.

3. We have $F^x \cap \mathrm{GSp}_{2g,\mathbb{Q}}(\phi_1) = T_F$, where T_F is a maximal torus of $\mathrm{GSp}_{2g,\mathbb{Q}}$ defined abstractly by the exact sequence

$$1 \longrightarrow T_F \xrightarrow{(\lambda,\mathrm{incl.})} \mathbb{Q}^x \times F^x \xrightarrow{\mathrm{incl.}/N_{F/F^+}} (F^+)^x \longrightarrow 1.$$

Here λ is the character $T_F \to \mathbb{Q}^x = \mathbb{G}_m$ induced by the standard similitude character $\mathrm{GSp}_{2g,\mathbb{Q}} \to \mathbb{G}_m$ and $F^+ \subset F$ is the totally real subfield. In other words, we have

$$T_F = \{x \in F^x \mid x \cdot {}^\tau x \in \mathbb{Q}^x\}$$

and the map $x \mapsto x \cdot {}^\tau x$ coincides with the similitude character.

3.67. On the other hand given a CM-field F of degree $2g$ and a $\delta \in F^-$, the bilinear form

$$\mathrm{tr}(x\delta({}^\tau y))$$

is symplectic, hence gives an identification $F \cong \mathbb{Q}^{2g}$ mapping this symplectic from to ϕ_1. The question is in this case whether a morphism $h : \mathbb{S} \to T_F$, given by a CM-type S, induces a morphism which makes ϕ_1 satisfy the conditions of a polarization (i.e. $\phi_1(\cdot, h(i)\cdot)$ definite) or, in other words, whether h can be identified with a point $\tau \in \mathbb{H}_g^\pm$?

Exercise 3.68. *h corresponds to a point $\tau \in \mathbb{H}_g^\pm$ if and only if all signs of the purely imaginary numbers*

$$\{\tau(\delta)\}_{\tau \in S}$$

are equal.

Obviously for all CM-types S a corresponding δ can be chosen (independence of the embeddings $F \to \mathbb{C}$) such that the condition holds true.

3.69. Assume that we have a maximal subtorus $T_F \subset \mathrm{GSp}_{2g}$ of the form described before and a morphism $h : \mathbb{S} \to T_F$ corresponding to a CM-type S such that the composition

$$\mathbb{S} \longrightarrow T_{F,\mathbb{R}} \hookrightarrow \mathrm{GSp}_{2g,\mathbb{R}}$$

corresponds to a $\tau \in \mathbb{H}_g^\pm$.
For each compact open subgroup $K \subset \mathrm{GSp}(\mathbb{A}_f)$ it induces a map

$$\iota : T_F(\mathbb{Q})\backslash\{h\} \times T_F(\mathbb{A}_f)/K \cap T_F(\mathbb{A}_f) \to \mathrm{GSp}(\mathbb{Q})\backslash\mathbb{H}_g^\pm \times \mathrm{GSp}(\mathbb{A}_f)/K = \mathcal{A}_{g,K}(\mathbb{C}) \tag{23}$$

Since $\mathcal{A}_{g,K}$ is defined over $\mathbb{Q}$, field automorphisms of $\mathbb{C}$ act on $\mathcal{A}_{g,K}(\mathbb{C})$.

Lemma 3.70. *1. Points in the image of ι are defined over number fields*

2. A Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ of a fixed field extension $E/\mathbb{Q}$ acts on the image for all compact open subgroups K.

[16] or equivalently F is a totally complex quadratic extension of a totally real subfield.

Proof. For an Abelian variety A with CM by F: $F \hookrightarrow \mathrm{Aut}(A)$ defined over $\mathbb{C}$ the CM-type is characterized by the isomorphism

$$\mathrm{Lie}(A) = \bigoplus_{s \in S} \mathbb{C}^{(s)}$$

of F-modules. For an automorphism τ of $\mathbb{C}$ we have

$$\mathrm{Lie}({}^{\tau}A) = \mathrm{Lie}(A) \otimes_{\mathbb{C},\tau} \mathbb{C} = \bigoplus_{s \in S} \mathbb{C}^{(\tau^{-1} \circ s)}$$

where ${}^{\tau}A$ is formed by applying τ to the coefficients of equations for A[17]. Therefore ${}^{\tau}A$ has CM by F with CM-type $\tau \circ S$. The field E is defined by

$$\mathrm{Gal}(E/\mathbb{Q}) = \{\tau \in \mathrm{Aut}(\mathbb{C}) \mid \tau \circ S = S\}.$$

To see that for a point $\iota([h,g])$ also the point ${}^{\tau}\iota([h,g]) =: ({}^{\tau}h, {}^{\tau}g)$ lies in the image of ι, we have to prove that also the morphism ${}^{\tau}h$ factors through the embedding $T_F \hookrightarrow \mathrm{GSp}_{2g}$ or a $\mathbb{Q}$-conjugate of it (the image of ι is the same when we change T_F by a conjugate torus T_F^{γ}). The actual morphism is then determined uniquely by the CM-type. One can show that the conjugacy class of T_F is determined by δ up to $\delta \mapsto \rho\delta$ with $\rho \in \mathbb{Q}^* \cdot N_{F|F^+}F^*$. This is also expressed by the fact that the conjugacy classes of embeddings of T_F (relative to a given embedding) are parametrized by $H^1(\mathbb{Q}, T_F) = (F^+)^*/\mathbb{Q}^* \cdot N_{F|F^+}F^*$. We have to show that the class of δ can be reconstructed from something invariant under the Galois action. We can choose an F-equivariant isomorphism

$$T(A) \cong \mathbb{A}_{F,f}$$

and the symplectic on $T(A)$ form becomes of the form

$$\mathrm{tr}(x(\delta_{\mathbb{A}})^{\tau} y)$$

Also here $\delta_{\mathbb{A}}$ is determined up to $\mathbb{A}_f^* N_{F/F^+}\mathbb{A}_{F,f}^*$. We have to show that the map

$$F^{+,*}/\mathbb{Q}^* N_{F/F^+F^*} \hookrightarrow \mathbb{A}_{F^+}^*/{\mathbb{A}_f}^* N_{F/F^+}\mathbb{A}_{F,f}^*$$

is injective. This is the case if the square marked 1 in the following diagram is Cartesian:

$$\begin{tikzcd}
T_F(\mathbb{Q}) \arrow[r] \arrow[d, hook] & F^x(\mathbb{Q})\mathbb{Q}^x(\mathbb{Q}) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "1"] & (F^+)^x(\mathbb{Q}) \arrow[r] \arrow[d, hook] & F^{+,*}/\mathbb{Q}^* N_{F/F^+}F^* \arrow[r] \arrow[d] & H^1(\mathbb{Q}, F^x) \times H^1(\mathbb{Q}, \mathbb{G}_m) = \{1\} \\
T_F(\mathbb{A}_f) \arrow[r] & F^x(\mathbb{A}_f)\mathbb{Q}^x(\mathbb{A}_f) \arrow[r] & (F^+)^x(\mathbb{A}_f) \arrow[r] & \mathbb{A}_{F^+}^*/{\mathbb{A}_f}^* N_{F/F^+}\mathbb{A}_{F,f}^* \arrow[r] & H^1(\mathbb{A}_f, F^x) \times H^1(\mathbb{A}_f, \mathbb{G}_m) = \{1\}
\end{tikzcd}$$

The H^1's are actually trivial here because of Hilbert 90 (for F^x it follows from Shapiro's Lemma). That the square 1 is Cartesian follows from the Hasse principle for norms in the quadratic extension F/F^+.
Now 2. is proven and 1. follows because the image of ι is finite. The action of $\mathrm{Aut}(\mathbb{C})$ on them consequently has to factor through a finite group. $\square$

The proof of the Lemma motivates

[17] In the language of schemes one can say that ${}^{\tau}A$ is defined by the following Cartesian square:

$$\begin{tikzcd}
{}^{\tau}A \arrow[r] \arrow[d] & A \arrow[d] \\
\mathrm{spec}(\mathbb{C}) \arrow[r, "\mathrm{spec}(\tau)"] & \mathrm{spec}(\mathbb{C})
\end{tikzcd}$$

and with the structural morphism to $\mathrm{spec}(\mathbb{C})$ given by the left vertical morphism.

Definition 3.71. *Let F be a CM-field with CM-type $S \subset \mathrm{Hom}(F, \mathbb{C})$. The subfield $E \subset \mathbb{C}$ determined by*

$$\mathrm{Gal}(E/\mathbb{Q}) = \{\tau \in \mathrm{Aut}(\mathbb{C}) \mid \tau \circ S = S\}$$

is called the **reflex field** *of* (F, S).

Actually the reflex field depends abstractly only on the torus T_F and the given morphism $h : \mathbb{S} \to T_F$. For remember $\mathbb{S}_\mathbb{C} \cong \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$. Let $\mu : \mathbb{G}_{m,\mathbb{C}} \to \mathbb{S}_\mathbb{C}$ be the inclusion of the first factor. E is precisely the field of definition of the morphism of $\mathbb{Q}$-tori: $h_\mathbb{C} \circ \mu : \mathbb{G}_{m,\mathbb{C}} \to T_{F,\mathbb{C}}$. This follows from the observation made in Lemma 3.62 that the decomposition $V^{-1,0} \oplus V^{0,-1}$, whose constituents are the weight spaces for -1 and 0, respectively, w.r.t. the morphism $h_\mathbb{C} \circ \mu$, is precisely the decomposition

$$\bigoplus_{s \in S} \mathbb{C}^{(s)} \oplus \bigoplus_{s \in \overline{S}} \mathbb{C}^{(s)}.$$

The field of definition of this decomposition is obviously E and hence E is the field of definition of the morphism $h_\mathbb{C} \circ \mu$.

3.72. The question remains whether we can explicitly describe the action of the Galois group on the CM Abelian varieties parametrized by the image of ι (23). It is easy to see that E is a CM field as well, in particular totally complex, hence class field theory gives an isomorphism

$$\mathrm{rec} : \mathrm{Gal}(E^{\mathrm{ab}}/E) \cong E^* \backslash \mathbb{A}_{E,f}^*.$$

In the case $g = 1$, we have $T_F = F^x$, $E = F$ and ι specializes to:

$$\iota : F^* \backslash \{h\} \times \mathbb{A}_{F,f}^* / K \cap \mathbb{A}_{F,f}^* \to \mathrm{GL}_2(\mathbb{Q}) \backslash \mathbb{H}_g^\pm \times \mathrm{GL}_2(\mathbb{A}_f)/K = \mathcal{A}_{1,K}(\mathbb{C}).$$

The *main theorem of complex multiplication for elliptic curves* states that we have simply:

$${}^\sigma\iota(h, \gamma) = \iota(h, \gamma \cdot \mathrm{rec}(\sigma)).$$

In particular the action is faithful in the limit over K, which may be translated to the fact that the coordinates of the points in the image of τ ([18]) generate the maximal Abelian extension of F. More generally, if $g > 1$, we do not have $T_F = E^x$ anymore but there is still a canonical morphism of tori, called **reflex norm**

$$N_S : E^x \to T_F$$

such that

$${}^\sigma\iota(h, \gamma) = \iota(h, \gamma \cdot N_S(\mathrm{rec}(\sigma))).$$

This is the *main theorem of complex multiplication for Abelian varieties.* On $\mathbb{Q}$-points, the reflex norm is explicitly given by

$$\begin{array}{rcl} N_S : E^* & \to & T_F(\mathbb{Q}) = \{\gamma \in F^* \mid {}^\tau\gamma\gamma \in \mathbb{Q}^*\} \\ x & \mapsto & \displaystyle\prod_{\sigma \in S^{-1}} {}^\sigma x \end{array}$$

Here S^{-1} is a CM-type of E derived from S, with reflex field $K \subset F$ (w.r.t. a fixed embedding $F \hookrightarrow \mathbb{C}$). The reflex norm is far from being injective, hence the maximal Abelian extension cannot be generated anymore by the points in the image of ι (by the way, not even by considering several CM-types with reflex field E at once).

[18] or — equivalently — the coefficients of the CM elliptic curve parametrized by h together with the coordinates of all its torsion points

3.73. Deligne gave an abstract definition of N_S which starts from the datum of an arbitrary $\mathbb{Q}$-torus T together with a morphism $h : \mathbb{S} \to T_{\mathbb{R}}$. As mentioned, h defines a field E which is the field of definition of the composition $h_{\mathbb{C}} \circ \mu$, i.e. there is a unique morphism

$$\mu_h : \mathbb{G}_{m,E} \to T \times_{\mathbb{Q}} E$$

such that $h_{\mathbb{C}} \circ \mu = \mu_h \times_E \mathbb{C}$. Since restriction of scalars is a functor, this defines a morphism of $\mathbb{Q}$-tori

$$\mathrm{Res}_{E/\mathbb{Q}}(\mu_h) : E^x \to \mathrm{Res}_{E/\mathbb{Q}} T$$

Since T is defined over $\mathbb{Q}$, there is a morphism "Norm"

$$N_{E/\mathbb{Q}} : \mathrm{Res}_{E/\mathbb{Q}} T \to T.$$

Definition 3.74. *The composition*

$$N_h := N_{E/\mathbb{Q}} \circ \mathrm{Res}_{E/\mathbb{Q}}(\mu_h) : E^x \to T$$

is a morphism of $\mathbb{Q}$-tori defined only in terms of the pair T, h. It is called the **reflex norm** *of the pair T, h.*

In the case $T = T_F \subset \mathrm{GSp}_{2g}$ the morphism N_h gives back the reflex norm N_S defined before.

3.75. It is instructive to translate the main theorem of complex multiplication for Abelian varieties in the above neat statement on moduli spaces back into a statement about the Abelian varieties themselves:

Theorem 3.76 (Main theorem of complex multiplication)**.** *Let A be a simple Abelian variety of dimension g and $\iota : F \hookrightarrow \mathrm{End}(A) \otimes \mathbb{Q}$ an embedding of a field of degree $2g$ over $\mathbb{Q}$ and let E be the reflex field of (A, ι). Let $\phi : A \to {}^tA$ be a polarization (which can be a $\mathbb{Q}$-isogeny). Then A is defined over $\overline{\mathbb{Q}}$ and for each $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/E)$ there is a $\mathbb{Q}$-isogeny α:*

$$A \xrightarrow{\alpha} {}^\sigma A$$

defined over $\overline{\mathbb{Q}}$ such that

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\mathbb{Q}\phi"'] & {}^\sigma A \arrow[d, "\mathbb{Q}^\sigma\phi"] \\ {}^tA \arrow[r, "{}^t\phi"] & {}^\sigma({}^tA) \end{tikzcd}$$

commutes up to $\mathbb{Q}$-scalar, and such that

$$\begin{tikzcd} V(A) \arrow[d, "N_S(\mathrm{rec}(\sigma))"'] \arrow[dr, "V(\sigma)"] & \\ V(A) \arrow[r, "V(\alpha)"'] & V({}^\sigma A) \end{tikzcd}$$

commutes (up to $T_F(\mathbb{Q})$), where N_S is the reflex norm (acting via $\iota \otimes_{\mathbb{Q}} \mathbb{A}_f$) and $\mathrm{rec} : \mathrm{Gal}(\overline{\mathbb{Q}}/E) \to \mathrm{Gal}(E^{\mathrm{ab}}/E) \cong \mathbb{A}^*_{E,f}$ *is the reciprocity morphism of class field theory.*

CHECK Sign!
Note that from the representability of the moduli problem and the equivalence with its non-adelic variant follows that A is actually defined over an Abelian extension of E itself.

3.77. We can anticipate the definition of canonical model which will be given for general Shimura varieties. Applied to the Shimura varieties associated with GSp parametrizing Abelian varieties it gives back the main theorem of complex multiplication, however, it seems more general. In fact, it is equivalent:

Theorem 3.78. *The model (cf. 3.60) given by the collection*

$$\{\mathcal{A}_{g,K}\}_K$$

(together with all the morphisms $J_{K,L}(g)\dots$) for all sufficiently small compact open subgroups $K \subset \mathrm{GSp}(\mathbb{A}_f)$ form a **canonical model** *of the Shimura variety*

$$\{\mathrm{GSp}_{2g}(\mathbb{Q})\backslash\mathbb{H}_g^{\pm} \times (\mathrm{GSp}_{2g}(\mathbb{A}_f)/K)\}_K$$

in the following sense:

- *For each $\mathbb{Q}$-torus $T \subset \mathrm{GSp}_{2g}$ such that we have a factorization*

$$h : \mathbb{S} \to T_{\mathbb{R}} \hookrightarrow \mathrm{GSp}_{2g,\mathbb{R}} \tag{24}$$

  for some h corresponding to a point in $\mathbb{H}_g^{\pm}$ and considering the induced map

$$\iota : T(\mathbb{Q})\backslash\{h\} \times T(\mathbb{A}_f)/(K \cap T(\mathbb{A}_f)) \to \mathrm{GSp}_{2g}(\mathbb{Q})\backslash\mathbb{H}_g^{\pm} \times \mathrm{GSp}_{2g}(\mathbb{A}_f)/K \cong \mathcal{A}_{g,K}(\mathbb{C}),$$

  we have that $\iota(h,\xi)$ is defined over $\overline{\mathbb{Q}}$ for all $\xi \in T(\mathbb{A}_f)$ and for each $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/E)$

$${}^{\sigma}\iota(h,\xi) = \iota(h, \xi \cdot N_h(\mathrm{rec}(\sigma))$$

  holds true, where N_h is the abstract reflex norm for the pair (T,h) (cf. 3.73).

We will later see that the condition in the theorem characterizes the model *uniquely* (up to isomorphism).

Proof. We have seen that for the special case of the maximal torus T_F this is nothing else than a reformulation of the main theorem of complex multiplication. It is not the case, however, that every $\mathbb{Q}$-torus of GSp_{2g} is of this form, even if a factorization (24) exists (cf. also the discussion in 3.63). However, we have the following statement:

- For each factorization as in (24) there exists a decomposition (symplectic linear isomorphism)

$$\kappa : \bigoplus_i \mathbb{Q}^{2g_i} \xrightarrow{\sim} \mathbb{Q}^{2g}$$

  into smaller symplectic spaces (where we choose the standard-forms ϕ_1 also on the $\mathbb{Q}^{2g_i}$) and CM-fields F_i of degree $2g_i$ over $\mathbb{Q}$ which induces an embedding:

$$\widetilde{\kappa} : \mathrm{GSp}_{2g_1} \times_{\mathbb{G}_m} \cdots \times_{\mathbb{G}_m} \mathrm{GSp}_{2g_n} \hookrightarrow \mathrm{GSp}_{2g}$$

  (in which the fiber products are formed w.r.t. the similitude character $\lambda : \mathrm{GSp}_{g_i} \to \mathbb{G}_m$) such that the following holds true: There exists morphisms $h_i : \mathbb{S} \to T_{F_i,\mathbb{R}}$ of the type considered in Lemma 3.66 such that the factorization (24) refines as follows[19]:

$$\begin{tikzcd}
& (T_{F_1} \times_{\mathbb{G}_m} \cdots \times_{\mathbb{G}_m} T_{F_n})_{\mathbb{R}} \arrow[dr, hook, "\widetilde{\kappa}"] & \\
\mathbb{S} \arrow[ur, "\prod h_i"] \arrow[dr] & & \mathrm{GSp}_{2g,\mathbb{R}} \\
& T_{\mathbb{R}} \arrow[ur, hook] &
\end{tikzcd} \tag{25}$$

Then h factors also through the intersection of the $\mathbb{Q}$-tori T and $\widetilde{\kappa}(T_{F_1} \times_{\mathbb{G}_m} \cdots \times_{\mathbb{G}_m} T_{F_n})$ and it is easy to see that the reflex field, the reflex norm, and the assertion of the theorem do depend on T only up to passing to a smaller or larger $\mathbb{Q}$-torus. Hence we can replace T by $\widetilde{\kappa}(T_{F_1} \times_{\mathbb{G}_m} \cdots \times_{\mathbb{G}_m} T_{F_n})$ and the assertion follows easily by induction on g and the main theorem of complex multiplication. $\square$

[19] unraveling the definitions, this is saying that the Abelian variety A defined by h is (isogenous to) a product of smaller Abelian varieties A_i, such that A_i is simple of dimension g_i, and has CM by F_i.

4 General Shimura varieties

4.1 Deligne's axioms

4.1. Let G be a reductive algebraic group defined over $\mathbb{Q}$ and $\mathbb{D} = G(\mathbb{R}) \cdot h_0 \subset \operatorname{Hom}(\mathbb{S}, G_{\mathbb{R}})$ be a conjugacy class.

Example 4.2. *In the case $G = \mathrm{GSp}_{2g}$, we have seen (cf. Lemma 3.17) that each complex structure J on $\mathbb{R}^{2g}$ which is compatible with the symplectic form ϕ_1 correspond to certain morphisms $h : \mathbb{S} \to \mathrm{GSp}_{2g}$. The set of all such complex structures such that the complex torus $(\mathbb{R}^{2g}, J)/\mathbb{Z}^{2g}$ is* polarized *w.r.t. ϕ_1 was denoted by $\mathbb{H}_g^{\pm}$ (Siegel's upper and lower half space). This gives an injection*

$$\mathbb{H}_g^{\pm} \hookrightarrow \operatorname{Hom}(\mathbb{S}, \mathrm{GSp}_{2g,\mathbb{R}})$$

and we have seen (cf. Corollary 3.19) that $\mathrm{GSp}_{2g}(\mathbb{R})$ acts transitively on $\mathbb{H}_g^{\pm}$ hence the image consists of one conjugacy class $\mathbb{D} \cong \mathbb{H}_g^{\pm}$.

In the general case Deligne gave the following axioms which imply, as we will see, that $\mathbb{D}$ is a finite union of *Hermitian symmetric domains*. We discussed them briefly in the introduction (Section 2.5) and will come back to the precise definition in the next section.

Definition 4.3. *The pair $(G, \mathbb{D})$, as in 4.1 is called* **a Shimura datum** *if*

(SV1) $\mathrm{Ad} \circ h$ *induces a representation of $\mathbb{S}$ on $\mathrm{Lie}(G_{\mathbb{C}})$ with the weights $(-1, 1), (0, 0)$ and $(1, -1)$ for some (hence all) $h \in \mathbb{D}$;*

(SV2) $\mathrm{int}(h(i))$ *is a Cartan involution of $G_{\mathbb{R}}^{\mathrm{ad}}$ for some (hence all) $h \in \mathbb{D}$;*

(SV3) G^{ad} has no factor H such that the projection of some (hence any) $h \in \mathbb{D}$ on $H_{\mathbb{R}}$ is trivial.

Under these conditions we define (following Deligne) the **Shimura variety** *associated with the datum $(G, \mathbb{D})$ and a compact open subgroup $K \subset G(\mathbb{A}_f)$ as the set of double cosets*

$$\mathrm{Sh}_K(G, \mathbb{D}) := G(\mathbb{Q}) \backslash \mathbb{D} \times G(\mathbb{A}_f)/K.$$

Remark 4.4. *For a reductive algebraic group G, defined over $\mathbb{R}$, an involutive automorphism $\sigma \in \mathrm{Aut}(G)$ is called a* **Cartan involution** *if*

$$G^{\sigma}(\mathbb{R}) = \{g \in G(\mathbb{C}) \mid \sigma(g) = \overline{g}\}$$

is compact. (These are actually the $\mathbb{R}$-points of another algebraic group G^{σ} which is the real form of G described by the cocycle in $H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathrm{Aut}(G_{\mathbb{C}}))$ defined by σ.) It is also a fact that $G^{\sigma}(\mathbb{R}) \cap G(\mathbb{R}) =$ $\mathrm{Stab}(\sigma, G(\mathbb{R}))$ *is* maximal *compact in G. Every* reductive *group G, defined over $\mathbb{R}$, has a Cartan involution, and all of them are conjugated. This is related to the fact that G has a* unique compact *real form G^c which can be obtained from any other real form G' by means of the respective Cartain involution on G'.*
The notion also makes sense for a real Lie group. In this case we say that an involution is a Cartan involution, if the stabilizer group is maximal *compact. If G is a reductive real algebraic group then σ is a Cartan involution of the Lie group $G(\mathbb{R})$ in this sense, if and only if it is a Cartan involution of G in the previous sense.*

4.5. We will see that $\mathrm{Sh}_K(G, \mathbb{D})$ is a finite union of quotients of the form

$$\Gamma \backslash \mathbb{D}^+$$

where $\mathbb{D}^+$ is a connected component of $\mathbb{D}$ (i.e. a Hermitian symmetric domain) and Γ is a congruence subgroup in $G_{\mathbb{R}}$ (w.r.t. the rational structure given by its chosen $\mathbb{Q}$-form G). For the purpose of constructing reasonable models over number fields the adelic viewpoint is much more convenient. For example the field of definition of the models of the $\mathbb{D}^+/\Gamma$ would depend on Γ whereas it will turn out that the field of definition of $\mathrm{Sh}_K(G, \mathbb{D})$ does not depend on K.

Why is this so?

There are two differences:

1. the passage to a *reductive* group,
2. the passage to the adelic language.

Let us discuss the first point:
Every Hermitian symmetric domain $\mathbb{D}$ is homogenous under a connected *semi-simple* real Lie group $\mathcal{G} = \mathrm{Aut}(\mathbb{D})^0$. There exists always an algebraic group G, defined over $\mathbb{Q}$, such that $G(\mathbb{R})^0 = \mathcal{G}$. Naively we could hence take $\mathbb{D}, G$ as our datum and study manifolds of the form

$$\Gamma \backslash \mathbb{D}$$

for congruence subgroups $\Gamma \subset G_{\mathbb{R}}$ w.r.t. the rational structure given by G. In the end, this does differ only in a very subtle way from Deligne's definition. We will discuss in the following section that it follows automatically from the theory of Hermitian symmetric domains that $\mathbb{D}$ can be naturally identified with a conjugacy class of morphisms $\mathrm{Hom}(U_1, G_{\mathbb{R}})$ where U_1 is the unitary group of dimension 1 (complex numbers of absolute value 1 considered as algebraic group over $\mathbb{R}$). Furthermore the property that such a conjugacy class is associated with a Hermitian symmetric domain in this way can be expressed by axioms similar to (SV1–SV3). The difference between Deligne's definition and this naive definition is (apart from the adelic issue) therefore merely a passing from G to a reductive group $\widetilde{G}$ and from U_1 to $\mathbb{S}$:

$$\begin{tikzcd}
\mathbb{S} \arrow[r, "h"] \arrow[d, "\pi: z\mapsto \frac{z}{\bar{z}}"'] & \widetilde{G}_{\mathbb{R}} \arrow[d] \\
\mathbb{U}_1 \arrow[r, "u"] & G_{\mathbb{R}}
\end{tikzcd}$$

A (conjectural) deeper reason why this passage is necessary to have good models over number fields, is the following. Assume that a faithful representation ρ of G has been chosen. We expect that (in dependence of ρ) the model parametrizes algebraic objects, more precisely: motives — in analogy with the parametrizing of polarized Abelian varieties by the model of the Shimura variety discussed in the beginning of the lecture. The link between such hypothetical moduli spaces and the points of the Shimura variety is the Hodge structure of the motive. Such a Hodge structre can be described by a morphism $h : \mathbb{S} \to G_{\mathbb{R}}$, rather than by a morphism $u : U_1 \to G_{\mathbb{R}}$. Furthermore, many of the representations $h : \mathbb{S} \to \mathrm{GL}_n$ describing the Hodge structure of interesting motives do not factor through a semi-simple algebraic group defined over $\mathbb{Q}$ but rather only through a reductive one (as was the case for polarized Abelian varieties). We will discuss these conjectural moduli problems (hopefully) briefly in the end of the lecture.
Let us discuss the second point:
We have already seen a motivation for the adelic point of view: The morphism

$$\iota : T_F(\mathbb{Q}) \backslash \{h\} \times T_F(\mathbb{A}_f)/K \cap T_F(\mathbb{A}_f) \to \mathrm{GSp}(\mathbb{Q}) \backslash \mathbb{H}_g^{\pm} \times \mathrm{GSp}(\mathbb{A}_f)/K = \mathcal{A}_{g,K}(\mathbb{C}) \tag{26}$$

describing (certain) Abelian varieties with CM by F! We have seen in 3.70 that the image of this morphism is characterized by purely algebraic conditions involving only the reflex field E. Hence the image of the morphism was equivariant under the absolute Galois group of E independently of the choice of K.
Instead, we could have considered a non-adelic version, like:

$$\{h\} \times T_F(\mathbb{Z}/N\mathbb{Z}) \to \mathrm{GSp}(\mathbb{Z}) \backslash \mathbb{H}_g^{\pm} \times \mathrm{GSp}(\mathbb{Z}/N\mathbb{Z}) \tag{27}$$

If we try to characterize the image of this non-adelic map, it involves transcendental choices (for example the isomorphism class of $H^1(A, \mathbb{Z})$ as $\mathcal{O}_F$-module) and hence there is no reason for the Galois group to respect the image (and, in fact, it will not).

4.2 Hermitian symmetric domains

4.6. Let $\mathbb{D}$ be a connected real $\mathcal{C}^\infty$-manifold and let g be a **Riemannian metric**, given by a positive definite symmetric bilinear form

$$g_p : T_p(\mathbb{D}) \times T_p(\mathbb{D}) \to \mathbb{R}$$

at each point $p \in \mathbb{D}$ that varies smoothly ($\mathcal{C}^\infty$) with p.

The pair $(\mathbb{D}, g)$ is called a **Riemannian manifold**. It is in particular a *metric space*, where the distance function $d(x, y)$ is given by the infimum of lengths (w.r.t. the given metric) of all paths from x to y.

4.7. Let $(\mathbb{D}, g)$ be a Riemannian manifold which is *simply connected* and *complete* as a metric space. Then the following conditions are equivalent:

1. For all $p \in \mathbb{D}$ there exists an isometry $s_p : \mathbb{D} \to \mathbb{D}$ fixing p such that $T_p(s_p) = -1$.
2. $\nabla R = 0$, where R denotes the Riemann curvature tensor defined by the following equation for two vector fields X and Y:
$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$
(This is also equivalent to the sectional curvature[20] being invariant under parallel transport.)
3. $\mathbb{D} \cong G/K$ where $G = \mathrm{Aut}(\mathbb{D})^+$ is a connected real Lie group and K is a *compact* subgroup which is an open subgroup of the stabilizer of an involution $\sigma \in \mathrm{Aut}(G)$.

If these equivalent conditions are satisfied, $(\mathbb{D}, g)$ is called a **Riemannian symmetric space**.
For a datum as in 3., where G is *any* connected Lie group, $\mathbb{D} := G/K$ is automatically Riemannian manifold (producing a K-invariant scalar product on $T_0(\mathbb{D})$ by compactness of K and then translating it by the action of G), hence a Riemannian symmetric space. If p is the point fixed by K then the involution σ in 3. is given by $\sigma = \mathrm{int}(s_p)$ (however, s_p does not need to lie in the unity component! ...and consequently might be an outer automorphism of G).

4.8. For a Riemannian symmetric space furthermore the following are equivalent

1. $T_p(\mathbb{D})$ carries a complex structure J s.t.
$$\langle u, v\rangle := g_p(u, v) + i g_p(u, Jv)$$
is a Hermitian form for one (and hence for all) p.
2. There exists a map $u_p : U_1 \hookrightarrow K$ fixing p such that u_p defines on $T_p(\mathbb{D})$ a complex structure for one (and hence for all) p.

If these equivalent conditions are satisfied, $(\mathbb{D}, g)$ is called a **Hermitian symmetric space**. It also follows that $\mathbb{D}$ is actually a complex manifold (as opposed to just an almost complex manifold). We will see a proof of this in the sequel. Note also that, in particular, we have $u_p(-1) = s_p$ and so s_p does have to lie in the unity component of $\mathrm{Aut}(\mathbb{D})$.
One can show using the description as G/K and the map u_p that every Hermitian symmetric space decomposes as the product of simple (indecomposable) spaces

$$\mathbb{D} = \mathbb{D}_1 \times \cdots \times \mathbb{D}_n$$

such that the $\mathbb{D}_i$ correspond to the simple factors of G. Then $\mathbb{D}_i$ is either

1. *compact* with positive sectional curvature or
2. *equal to* $\mathbb{C}$ with zero curvature or

[20]The sectional curvature is defined for a 2-dimensional subspace $\langle e_0, e_1\rangle \subset T_p(\mathbb{D})$ by $g_p(R(e_0, e_1)e_0, e_1)$ where e_0, e_1 are orthonormal. It is independent of this choice of basis. The collection over all 2-dimensional subspaces determines R.

3. *non-compact* with negative sectional curvature.

The latter are the most interesting ones for us and are characterized as follows (we do not necessarily assume that they are simple):

4.9. Let $\mathbb{D} = G/K$ be a Hermitian symmetric space. Then the following are equivalent:

1. G is semi-simple and $K \subset G$ is *maximal* compact,
2. $\sigma = s_p = u_p(-1)$ is a Cartan involution,
3. $\mathbb{D}$ is a **Hermitian symmetric domain**, i.e. there is a holomorphic embedding

$$\mathbb{D} \hookrightarrow \mathbb{C}^n$$

   with *bounded* image,

4. The sectional curvature is negative.

4.3 Classification of Hermitian symmetric domains

The different characterizations of Hermitian symmetric spaces (in particular Hermitian symmetric domains) given in the last section, enable us to classify them in terms of group theoretical data, already very much in the spirit of Deligne's axioms:

Theorem 4.10. *There is a bijection*

$$\{\textit{Hermitian symmetric domains } \mathbb{D}\}_{/\sim} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{ll} & G \textit{ semi-simple adjoint connected Lie group} \\ & G \cdot u_0 \subset \mathrm{Hom}(U_1, G) \textit{ a conjugacy class such that} \\ (1) & \textit{in the representation } \mathrm{Ad} \circ u_0 : U_1 \to \mathrm{GL}(\mathrm{Lie}(G)) \\ & \textit{only the characters } z^{-1}, 1, z \textit{ occur,} \\ (2) & \mathrm{Ad}(u_0(-1)) \textit{ is a Cartan involution of } G, \\ (3) & G \textit{ has no factor on which projection of } u_0 \textit{ is trivial.} \end{array} \right\}_{/\sim}$$

Sketch of proof. Let a Hermitian symmetric domain $\mathbb{D}$ be given. By property 3. of 4.7 we know that $\mathbb{D} = G/K$, where $G = \mathrm{Aut}(\mathbb{D})^+$ is a connected Lie group and K is an open subgroup of the stabilizer of $u_0(-1)$. We get a map

$$\mathbb{D} \to \mathrm{Hom}(U_1, G) \tag{28}$$

by associating with any $p \in \mathbb{D}$ the well-defined morphism u_p fixing p and acting on the tangent space as multiplication by z. This map is G-equivariant and hence the image is the conjugacy class of one of the morphisms, say u_0. Let K' be the stabilizer of a morphism u_0 under conjugation. We have $K \subset K' \subset \mathrm{Stab}(u_0(-1))$ and the composed embedding is open and $\mathrm{Stab}(u_0(-1))$ is compact. But K' is connected because all centralizers of a compact torus in a semi-simple Lie group are connected. Therefore $K = K'$. Therefore the map (28) is injective. Furthermore K as stabilizer of the morphism u_0 contains the center of G. Hence the center acts trivially on $\mathbb{D}$ and so must be trivial. Hence G is adjoint. Hence we associated with $\mathbb{D}$ an adjoint semi-simple group connected Lie group G and a conjugacy class $G \cdot u_0 \in \mathrm{Hom}(U_1, G)$. We have to verify that this datum satisfies the properties (1–3):
(1) follows because $\mathrm{Lie}(K)$ is the fixed subspace under the action of $\mathrm{Ad} \circ u_0$ on $\mathrm{Lie}(G)$. Choosing an equivariant complement:

$$\mathrm{Lie}(G) = \mathrm{Lie}(K) \oplus \mathfrak{p}$$

we see that $\mathfrak{p} \cong T_0(\mathbb{D})$. Since u_0 defines a complex structure on $T_0(\mathbb{D})$, we get that $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}^{(-1)} \oplus \mathfrak{p}^{(1)}$ where $u_0(z)$ acts as z^{-1} and z, respectively.
(2) was said in 4.9.
(3) By (1) that factor would have to be contained in K and hence would act trivially on $\mathbb{D}$ contradicting $G = \mathrm{Aut}(\mathbb{D})^+$.

Conversely consider a pair of a real semi-simple adjoint connected Lie group, and a conjugacy class $G \cdot u_0 \in \mathrm{Hom}(U_1, G)$ satisfying (1–3). Let K be the centralizer of u_0. Hence the conjugacy class is isomorphic to $\mathbb{D} := G/K$. Since $\mathrm{Ad}(u_0(-1))$ is a Cartan involution, we have $\mathrm{Stab}(u_0(-1))$ is compact and hence

$$K \subset \mathrm{Stab}(u_0(-1))$$

is compact. Again $\mathrm{Lie}(K)$ is the subspace fixed by $u_0(z)$ and if we chose a complement

$$\mathrm{Lie}(G) = \mathrm{Lie}(K) \oplus \mathfrak{p}$$

(1) implies that $\mathrm{Ad}(u_0(z))$ give a complex structure on $\mathfrak{p} \cong T_0(\mathbb{D})$. Furthermore $\mathrm{Lie}(K) = \mathrm{Stab}(u_0(-1))$ hence K is open in $\mathrm{Stab}(u_0(-1))$. Therefore $\mathbb{D}$ satisfies one of the characterizations of Hermitian symmetric domain.
Consider the *surjective* morphism $G \to \mathrm{Aut}(\mathbb{D})^+$. Its kernel would be a factor of G because G is semi-simple adjoint and since it stabilizes u_0 the projection of u_0 would be trivial on it. Hence we see that $G \to \mathrm{Aut}(\mathbb{D})^+$ is an isomorphism and hence the two associations are inverse to each other. □

Note that a factor of G, on which the projection of u_0 is trivial, would be automatically compact (as contained in K) and does not change G/K. Later it will be convenient and important to allow such factors as long as it is not also a factor w.r.t. the chosen $\mathbb{Q}$-structure.

4.4 The passage from U_1 to $\mathbb{S}$.

4.11. Since U_1 is compact, representations of U_1 as a Lie group are all algebraic. That is, a morphism

$$U_1 \to \mathrm{GL}_n(\mathbb{R})$$

of Lie groups always comes from a homomorphism of algebraic groups

$$\mathbb{U}_1 \to \mathrm{GL}_{n,\mathbb{R}}$$

where $\mathbb{U}_1 \subset \mathbb{S}$ is the subtorus of norm 1 elements. We have seen that representations of $\mathbb{S}$ on a real vector space are the same as Hodge structure via the weight spaces

$$V^{p,q} = \{v \in V_{\mathbb{C}} \mid h(z)v = z^{-p}\overline{z}^{-q}v\}.$$

Similarly, representations of $\mathbb{U}_1$ are the same as decompositions

$$V_{\mathbb{C}} = \bigoplus_p V^p \quad \text{with} \quad \overline{V^p} = V^{-p} \tag{29}$$

by means of

$$V^n = \{v \in V_{\mathbb{C}} \mid u(z)v = z^{-n}v \quad (= \overline{z}^n v)\}.$$

Each Hodge structure $\{V^{p,q}\}$ determines a structure (29) by setting $V^n = \sum_{p-q=n} V^{p,q}$. On the level of tori this corresponds to composition with the inclusion $\mathbb{U}_1 \subset \mathbb{S}$.

4.12. Note that there is also a morphism

$$\begin{array}{rcl} \pi : \mathbb{S} & \to & \mathbb{U}_1 \\ z & \mapsto & \dfrac{z}{\overline{z}} \end{array}$$

On the level of representations this corresponds to sending a structure (29) given by $\{V^n\}$ to the following Hodge structure (of weight 0)

$$V^{p,q} = \begin{cases} V^p & \text{if } p+q=0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence this just describes the full embedding of Hodge structure of weight 0 into all Hodge structures. Note that the two functors

$$\{\text{Hodge structures of weight } 0\} \leftrightarrow \{\text{Hodge structures}\}$$

so described are *not* inverse to each other in neither direction.

4.13. Given a representations $\rho : G \to \mathrm{GL}_n$ a conjugacy class

$$\mathbb{D} = G(\mathbb{R})u_0 \subset \mathrm{Hom}(\mathbb{U}_1, G_\mathbb{R}) \quad \text{resp.} \quad \mathbb{D} = G(\mathbb{R})h_0 \subset \mathrm{Hom}(\mathbb{S}, G_\mathbb{R})$$

gives rise (via composition with ρ) to a family of structures (29) (or, equivalently, Hodge structures of weight 0) and arbitrary Hodge structures, respectively on $\mathbb{D}$. Deligne's insight was, that it is necessary to pass from $\mathbb{U}_1$ to $\mathbb{S}$ to get also Hodge structures of other weights into the picture, which are naturally associated with families of abgebraic objects parametrized by (quotients of) $\mathbb{D}$.

4.14. Given a Hermitian symmetric domain, we have the canonical conjugacy class $\mathbb{D} = G(\mathbb{R})u_0 \subset \mathrm{Hom}(\mathbb{U}_1, G_\mathbb{R})$, where $G_\mathbb{R}$ is a real algebraic group with $G(\mathbb{R})^+ = \mathrm{Aut}(\mathbb{D})^+$. We automatically get a conjugacy class $G(\mathbb{R})h_0 \subset \mathrm{Hom}(\mathbb{S}, G_\mathbb{R})$ (isomorphic to $\mathbb{D}$ as a set) by means of composition with the morphism $\mathbb{S} \to \mathbb{U}_1$. If u_0 satisfies the properties (1–3) of (4.10), it is clear that $h_0 = u_0 \circ \pi$ satisfies the properties (SV1–SV3).

4.15. From a conjugacy $G(\mathbb{R})h_0 \subset \mathrm{Hom}(\mathbb{S}, G_\mathbb{R})$, where G is now merely a *reductive* group, satisfying axioms (SV1–SV3), we get back a datum like in 4.10 as follows: Consider the composition in the first line of

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r] & \mathbb{S} \arrow[r, "h_0"] \arrow[d, "\pi"'] & G_\mathbb{R} \arrow[r, two heads] & G_\mathbb{R}^{\mathrm{ad}} \\
& \mathbb{U}_1 \arrow[rru, dotted, "u_0"'] & &
\end{tikzcd}$$

From axiom (SV2) follows that the composition $\mathbb{G}_m \to G_\mathbb{R}^{\mathrm{ad}}$ is trivial because it induces the trivial action on $\mathrm{Lie}(G_\mathbb{R})$ and the adjoint representation of $G_\mathbb{R}^{\mathrm{ad}}$ on $\mathrm{Lie}(G_\mathbb{R})$ is faithful. Therefore the morphism $\mathbb{S} \to G_\mathbb{R}^{\mathrm{ad}}$ factors through π. Furthermore the resulting morphism u_0 satisfies the properties (1–3).
The point is that this does not describe a one-to-one correspondence. There are different diagrams like above inducing the same $(G_\mathbb{R}^{\mathrm{ad}}, u_0)$. However, the above procedure describes an isomorphism

$$G(\mathbb{R})^+ \cdot h_0 \xrightarrow{\sim} G^{\mathrm{ad}}(\mathbb{R})^+ \cdot u_0.$$

This shows that $G(\mathbb{R}) \cdot h_0$ has the structure of a union of finitely many Hermitian symmetric domains.

Example 4.16. *Consider the upper half plane $\mathbb{H}$. The real algebraic group PGL_2 satisfies $\mathrm{Aut}(\mathbb{H})^+ = \mathrm{PGL}_2(\mathbb{R})^+$. The morphism*

$$\begin{array}{rcl} u_0 : \mathbb{U}_1 & \to & \mathrm{PGL}_{2,\mathbb{R}}^+ \\ a + bi & \mapsto & \sqrt{\begin{pmatrix} a & b \\ -b & a \end{pmatrix}} \end{array}$$

fixes i and acts on the tangent space $T_i(\mathbb{H})$ by multiplication by $z = a+bi$. The pair $(\mathrm{PGL}_2(\mathbb{R})^+, \mathrm{PGL}_2^+(\mathbb{R}) \cdot u_0)$ is therefore the pair associated with $\mathbb{H}$ (under the correspondence of Theorem 4.10). Note that this defines a valid map of Lie groups because $\{\pm 1\}$ is mapped to 1 in PGL_2.
We have the following extensions where the first is given by the procedure 4.14 above

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r] & \mathbb{S} \arrow[r, "h_0"] \arrow[d, "\pi"'] & \mathrm{PGL}_{2,\mathbb{R}} \arrow[r, equal] & \mathrm{PGL}_{2,\mathbb{R}}^{\mathrm{ad}} \\
& \mathbb{U}_1 \arrow[rru, dotted, "u_0"'] & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r] & \mathbb{S} \arrow[r, "\widetilde{h}_0"] \arrow[d, "\pi"'] & \mathrm{GL}_{2,\mathbb{R}} \arrow[r, two heads] & \mathrm{GL}_{2,\mathbb{R}}^{\mathrm{ad}} = \mathrm{PGL}_{2,\mathbb{R}} \\
& \mathbb{U}_1 \arrow[rru, dotted, "u_0"'] & &
\end{tikzcd}$$

where $\widetilde{h}_0$ is the morphism

$$\begin{array}{rcl} \widetilde{h}_0 : \mathbb{S} & \to & \mathrm{GL}_{2,\mathbb{R}} \\ a+bi & \mapsto & \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \end{array}$$

(this gives another justification that u_0 above is a morphism of algebraic groups).
Hence the pairs $(\mathrm{GL}_2, \mathrm{GL}_2(\mathbb{R}) \cdot \widetilde{h}_0)$ and $(\mathrm{PGL}_2, \mathrm{PGL}_2(\mathbb{R}) \cdot h_0)$ are two 'lifts' of the same datum. They induce different families of Hodge structures on $\mathbb{H}^{\pm}$. For a representation $\rho : \mathrm{GL}_{2,\mathbb{R}} \to \mathrm{GL}_N$ which factors through $\mathrm{PGL}_2(\mathbb{R})$, we get the same family, which is of always of weight 0. However, only the standard representation of $\mathrm{GL}_{2,\mathbb{R}}$ induces the family of Hodge structures that we considered in the beginning of the lecture (complex structures giving elliptic curves). It is of weight -1 and the representation does not factor through $\mathrm{PGL}_{2,\mathbb{R}}$.
Note that there is no diagram in which SL_2 occurs:

$$\begin{tikzcd} \mathbb{G}_m \arrow[r] & \mathbb{S} \arrow[r, "/"] \arrow[d, "\pi"'] & \mathrm{SL}_{2,\mathbb{R}} \arrow[r, two heads] & \mathrm{PGL}_{2,\mathbb{R}} \\ & \mathbb{U}_1 \arrow[urr, dotted, "u_0"'] & & \end{tikzcd}$$

Conclusion:

Theorem 4.17. *There is a surjection with canonical splitting*

$$\left\{ \begin{array}{ll} & G \text{ real reductive algebraic group} \\ & G(\mathbb{R}) \cdot h_0 \in \mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}}) \text{ a conjugacy class such that} \\ (1) & \text{the representation } \mathrm{Ad} \circ h_0 : \mathbb{S} \to \mathrm{GL}(\mathrm{Lie}(G)) \\ & \text{is of type } (-1,1),(0,0),(1,-1), \\ (2) & \mathrm{Ad}(h_0(i)) \text{ is a Cartan involution of } G^{\mathrm{ad}}(\mathbb{R}), \\ (3) & G^{\mathrm{ad}}_{\mathbb{R}} \text{ has no factor on which projection of } h_0 \text{ is trivial.} \end{array} \right\} /\sim$$

$$\longrightarrow \left\{ \begin{array}{ll} & G \text{ semi-simple adjoint connected Lie group} \\ & G \cdot u_0 \in \mathrm{Hom}(U_1, G) \text{ a conjugacy class such that} \\ (1) & \text{in the representation } \mathrm{Ad} \circ u_0 : U_1 \to \mathrm{GL}(\mathrm{Lie}(G)) \\ & \text{only the characters } z^{-1}, 1, z \text{ occur,} \\ (2) & \mathrm{Ad}(u_0(-1)) \text{ is a Cartan involution of } G, \\ (3) & G \text{ has no factor on which projection of } u_0 \text{ is trivial.} \end{array} \right\} /\sim$$

The only difference between the left hand side and Deligne's definition is that a $\mathbb{Q}$-structure of G has to be chosen[21]. One both sides this is necessary to define interesting arithmetic quotients of $\mathbb{D}$.

Example 4.18. *We investigate how the different "lifts" of the data giving the Hermitian symmetric domain $\mathbb{H}$ to a (real) Shimura datum (in the sense of Deligne) that we got in 4.16 correspond to different moduli problems.*

$$\mathrm{GL}_2(\mathbb{Q}) \backslash \mathbb{H}^{\pm} \times \mathrm{GL}_2(\mathbb{A}_f)/K(N, \mathrm{GL}_2) \cong \mathrm{GL}_2(\mathbb{Z}) \backslash \mathbb{H}^{\pm} \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$$

is in bijection with isomorphism classes of elliptic curves E with isomorphism $\xi : (\mathbb{Z}/N\mathbb{Z})^2 \to E[N]$. Note that several connected components occur naturally.

$$\mathrm{PGL}_2(\mathbb{Q}) \backslash \mathbb{H}^{\pm} \times \mathrm{PGL}_2(\mathbb{A}_f)/K(N, \mathrm{PGL}_2) \cong \mathrm{PGL}_2(\mathbb{Z}) \backslash \mathbb{H}^{\pm} \times \mathrm{PGL}_2(\mathbb{Z}/N\mathbb{Z})$$

is in bijection with isomorphism classes of elliptic curves E with isomorphism $\xi : (\mathbb{Z}/N\mathbb{Z})^2 \to E[N]$ modulo the action of $(\mathbb{Z}/\mathbb{N}\mathbb{Z})^*$. The attentive reader might have noticed that this moduli problem seems to be associated still with a family of Hodge structures of weight -1. This is however not quite true because a Hodge structure gets modded out by the automorphisms $\{\pm 1\}$. This gets repaired, for instance, if we map the Hodge structure (complex structure) $V = (\mathbb{R}^2, \tau)$ to its symmetric square twisted by one: $(\mathrm{Sym}^2 V)(1)$. It is of weight zero. This corresponds to taking the symmetric square representation of GL_2 multiplied with $\det^{-1}$. It factors through PGL_2.

[21] And axiom (3) is slightly modified, taking the $\mathbb{Q}$-structure into account.

Also to the set

$$\mathrm{SL}_2(\mathbb{Q})\backslash\mathbb{H} \times \mathrm{SL}_2(\mathbb{A}_f)/K(N, \mathrm{SL}_2) \cong \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \times \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$$

can be associated a moduli problem (although $\mathbb{H}$ does not have a natural interpretation as conjugacy class of morphisms in $\mathrm{Hom}(\mathbb{S}, \mathrm{SL}_{2,\mathbb{R}})$) which is, however, only defined over $\mathbb{Q}(\zeta_N) \subset \mathbb{C}$, hence its field of definition depends on $K = K(N, \mathrm{SL}_2)$. It is given by elliptic curves E with isomorphism $\xi : (\mathbb{Z}/N\mathbb{Z})^2 \to E[N]$ which is "of determinant 1" in the sense that it maps the standard symplectic form to the Weil paring. This involves a trivialization of $\mathbb{G}_m[N](\mathbb{C}) \cong (\mathbb{Z}/N\mathbb{Z})^*$ hence the choice of a primitive N-th root of unity in $\mathbb{C}$.

4.5 The adelic description

The second fundamental difference to a naive consideration of quotients $\Gamma\backslash\mathbb{D}^+$ is the passage to the adelic language. We will investigate briefly the difference: Firstly for the adelic quotient, we have

Lemma 4.19.

$$G(\mathbb{Q})\backslash\mathbb{D} \times G(\mathbb{A}_f)/K = \bigcup_{[\gamma]} \Gamma_\gamma\backslash\mathbb{D} \tag{30}$$

Here $[\gamma]$ runs through the classes of G w.r.t. K, i.e. through the double cosets in

$$G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$$

(the so-called **set of classes** *of G w.r.t. K) and*

$$\Gamma_\gamma = \gamma K \gamma^{-1} \cap G(\mathbb{Q}).$$

The groups Γ_γ do depend on the choice of representative $\gamma \in [\gamma]$, but are conjugated for different choices. Hence the map and the decomposition (30) are essentially independent of the choice of representatives. It is a fundamental fact that the set of classes $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$ w.r.t. any compact open subgroup is *finite* for any algebraic group defined over $\mathbb{Q}$. This encodes all sorts of "finiteness of class number" results for fields, quadratic forms, Hermitian forms, etc.

Proof. The map (from right to left) is given by sending a $\tau \in \Gamma_\gamma\backslash\mathbb{D}$ to the double coset $[\tau, \gamma]$. This is obviously a bijection. $\square$

Since $\mathbb{D}$ itself is a union of copies of a Hermitian symmetric domain $\mathbb{D}^+$, this establishes that

$$G(\mathbb{Q})\backslash\mathbb{D} \times G(\mathbb{A}_f)/K = \bigcup \Gamma_\gamma^+\backslash\mathbb{D}^+$$

with an appropriate union and setting $\Gamma_\gamma^+ := \Gamma_\gamma \cap \mathrm{Stab}_{G(\mathbb{R})}(\mathbb{D}^+)$.

Remark 4.20. *It is a famous theorem (cf. [10]) that, if G^{der} is simply connected, then the class number of G^{der} is one w.r.t. any compact open subgroup. In this case the connected components can be described in terms of the classes of the torus $T = G/G^{\mathrm{der}}$ which are a more accessible invariant. In particular the set of classes of T w.r.t. any compact open subgroup form an Abelian group. For example, if T is of the form $\mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m$ they are just given by the class group of K and extensions of it. We refer to [8, Theorem 5.17] for details of this.*

4.6 Arbitrary Shimura varieties as parameter spaces of Hodge structures

4.21. We have seen two instances of what it means for a Hodge structure parametrized by a morphism $h : \mathbb{S} \to \mathrm{GL}_N$ to factor through a subgroup $G \subset \mathrm{GL}_{N,\mathbb{Q}}$ (base changed to $\mathbb{R}$), that is, through a faithful respresentation. The first, $\mathrm{GSp}_{2g} \subset \mathrm{GL}_{N,\mathbb{Q}}$, was related to polarizations of Abelian varieties (cf. 3.17), and the second, $F^x \subset \mathrm{GL}_{N,\mathbb{Q}}$, to CM of Abelian varieties (cf. 3.62). In both cases one interpretation was that certain tensors (bilinear forms or endomorphisms) are of a certain Hodge type. We want to generalize this relation.

4.22. Let G be a reductive group defined over $\mathbb{Q}$ and $\rho : G \hookrightarrow \mathrm{GL}_{N,\mathbb{Q}}$ a faithful representation. Given a conjugacy class $\mathbb{D} \subset \mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$, as in a Shimura datum, by composition with ρ we get a family of Hodge structures on $\mathbb{R}^N$ parametrized by $\mathbb{D}$. Those obviously all have the property above.

Lemma 4.23. *Let V be a $\mathbb{Q}$-vector space. Given any algebraic subgroup $G \subset \mathrm{GL}_{\mathbb{Q}}(V)$ one can find a finite set of tensors*

$$t_1, \ldots, t_n \in V^{\otimes} = \bigoplus_{p,q} V^{\otimes p} \otimes (V^*)^{\otimes q}$$

such that

$$G = \{g \in \mathrm{GL}_N(V) \mid g \cdot t_i = t_i \quad \forall i\}.$$

Then we have

Definition 4.24. *A Hodge structure on $\mathbb{Q}^N$ (associated with a morphism $h : \mathbb{S} \to \mathrm{GL}_{N,\mathbb{R}}$) is called* **compatible with the representation** $\rho : G_{\mathbb{Q}} \hookrightarrow \mathrm{GL}_{N,\mathbb{Q}}$ *if the following equivalent conditions hold*[22]

1. *h factors through ρ,*
2. *$t_i \in ((\mathbb{Q}^N)^{\otimes})^{0,0}$ for all i.*

It is often the case that the parametrized Hodge structures live on some vector space V (as e.g. $H_1(A, \mathbb{Q})$ that was considered in chapter 3) instead of the fixed vector space $\mathbb{Q}^N$. These vector spaces carry the same kind of structure which is chosen on $\mathbb{Q}^N$ (as e.g. a symplectic form up to scalar that was considered in chapter 3). To formulate this purely group-theoretically we define:

Definition 4.25. *Let G be an algebraic group defined over $\mathbb{Q}$ with faithful representation $\rho : G_{\mathbb{Q}} \hookrightarrow \mathrm{GL}_{N,\mathbb{Q}}$. A G-**structure** on a vector space V is a class*

$$G(\mathbb{Q}) \cdot \alpha : V \to \mathbb{Q}^N$$

of isomorphisms of V with $\mathbb{Q}^N$ modulo $G(\mathbb{Q})$ acting by post-composition. There is an obvious notion of a morphism of G-structures such that $G(\mathbb{Q})$ occurs as the group of automorphisms of any such structure.

If G is described inside $\mathrm{GL}_{N,\mathbb{Q}}$ by a set of tensors $t_i \in (\mathbb{Q}^N)^{\otimes}$, it is the same to give a G-structure on a vector space V or a set of tensors $\widetilde{t_i} \in V^{\otimes}$ such that there exists an isomorphism

$$\alpha : V \to \mathbb{Q}^N$$

such that $\alpha(\widetilde{t_i}) = t_i$. The class of all such isomorphisms is then the corresponding G-structure.

Example 4.26.

1. *A GSp_{2g}-structure (w.r.t. the standard representation of GSp_{2g}) on V is the same as the datum of a symplectic form on V up to scalar.*
2. *A F^x-structure (w.r.t. the standard representation of F^x) is the same as a F-vector space structure on V.*
3. *A $\mathrm{SO}(F)$-structure on V (w.r.t. the standard representation), where $F \in M_N(\mathbb{Q})$ is a symmetric matrix, is the same as giving a symmetric bilinear form B on V such that there is an isometry*

$$(\mathbb{Q}^N, F) \cong (V, B).$$

Finally, we define:

Definition 4.27. *A Hodge structure on a vector space V and a G-structure $G(\mathbb{Q}) \cdot \alpha : V \to \mathbb{Q}^N$ are called compatible if the following equivalent conditions hold:*

[22] To get also different Hodge types (n,n) into the picture, one should generalize this as follows: Consider a subgroup $\widetilde{G} \subset \mathrm{GL}_N$ (e.g. $\widetilde{G} = \mathrm{GSp}$) with a weight morphism $\widetilde{G} \to \mathbb{G}_m$ and consider $V^{\otimes} = \bigoplus_{p,q,n} V^{\otimes p} \otimes (V^*)^{\otimes q} \otimes \mathbb{Q}(n)$ where $\mathbb{Q}(n)$ is the representation of $\widetilde{G}$ factoring through the weight n representation of $\mathbb{G}_m$. Consider then only h which factor through $\widetilde{G}$ and such that the composition $\mathbb{G}_m \hookrightarrow \mathbb{S} \to \widetilde{G}_{\mathbb{R}} \to \mathbb{G}_m$ is the identity.

1. *h factors through $({}^{\alpha}\rho)_{\mathbb{R}} : G \to \mathrm{GL}(V)$,*
2. *$\widetilde{t}_i \in (V^{\otimes})^{0,0}$ for all i.*

The notion is quite trivial so far, however becomes useful in families: For a complex analytic manifold M a **$\mathbb{Q}$-local system with G-structure** is a $\mathbb{Q}$-local system $\mathcal{L}$ with a class in the quotient sheaf:

$$\alpha \in \mathcal{ISO}(\mathcal{L}, (\mathbb{Q}^N)_M)/G(\mathbb{Q})$$

where $\mathcal{ISO}(\mathcal{L}, (\mathbb{Q}^N)_M)$ is the sheaf of isomorphisms. This means explicitly that we have isomorphisms (trivializations) on a cover $\{U_i\}$ of M:

$$\alpha_i : \mathcal{L}|_{U_i} \to (\mathbb{Q}^N)_{U_i}$$

whose glueing condition has hold only modulo $G(\mathbb{Q})$. For example, having a G-structure for the trivial group implies that the local system is trivial, whereas a G-structure for $G = \mathrm{GL}_N$ is not a datum at all.

4.28. We have seen how we can interpret the Hermitian symmetric domain $\mathbb{D}$ as a set of Hodge structures. This, so far, did not take the complex structure on $\mathbb{D}$ into account. We will see that this set of Hodge structure forms *a family of Hodge structures* on the complex analytic manifold $\mathbb{D}$:

Definition 4.29. *Let B be a complex analytic manifold. A set of Hodge structures on a vector space V (over $\mathbb{Q}$ or $\mathbb{R}$)*

$$b \mapsto \{V_b^{p,q}\} \qquad V_{\mathbb{C}} = \bigoplus V_b^{p,q}$$

parametrized by B is called a **family of Hodge structures** *if the associated* **Hodge filtrations**

$$(F^p V_{\mathbb{C}})_b = \bigoplus_{i \geq p} V_b^{i,j}$$

vary holomorphically, i.e. form a filtration by holomorphic vector bundles of the trivial bundle $V \otimes_{\mathbb{Q}} \mathcal{O}_B$.

4.30. Recall that a Hodge structure is called of **of weight** n if the $V^{p,q}$ are zero unless $p + q = n$, or, equivalently, if it has the property that the composition $h \circ w : \mathbb{G}_m \hookrightarrow \mathbb{S} \to \mathrm{GL}_{N,\mathbb{R}}$ is given by $h(w(z))v = z^n v$ for all $v \in \mathbb{R}^N$. For a Hodge structure of weight n, we can reconstruct the Hodge structure from its associated Hodge filtration by

$$V^{p,q} = \begin{cases} F^p \cap \overline{F^q} & p + q = n, \\ 0 & \text{otherwise.} \end{cases}$$

Consider again a reductive group G, defined over $\mathbb{Q}$, and a conjugacy class $\mathbb{D} = G(\mathbb{R}) \cdot h_0 \subset \mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$ satisfying (SV1–3), with a faithful representation $\rho : G \hookrightarrow \mathrm{GL}_{N,\mathbb{Q}}$. Assume for simplicity that all Hodge structures given by the $h \in \mathbb{D}$ have weight n, i.e. assume that the composition $\mathbb{G}_m \hookrightarrow \mathbb{S} \to G_{\mathbb{R}} \hookrightarrow \mathrm{GL}_{N,\mathbb{R}}$ is central. Consider the functor

$$\mathcal{H}(G, \mathbb{D})(B) = \left\{ \begin{array}{c} \cdots \subset \mathcal{F}^{p+1} \subset \mathcal{F}^p \subset \cdots \subset V \otimes_{\mathbb{Q}} \mathcal{O}_B \\ \text{filtration by holomorphic subsheaves,} \\ \text{such that for all } b \in B: \\ \{\mathcal{F}_b^p\} \text{is a Hodge filtration of weight } n \text{ such that the associated} \\ \text{morphism } h_b : \mathbb{S} \to \mathrm{GL}_{N,\mathbb{R}} \text{ factors through } G_{\mathbb{R}} \text{ and lies in } \mathbb{D}. \end{array} \right\}$$

The fact that the morphism $h_b : \mathbb{S} \to \mathrm{GL}_{N,\mathbb{R}}$ factors through $G_{\mathbb{R}}$ is saying precisely that the Hodge structure is *compatible with the representation* in the sense of the Definition 4.24. Hence it could equally well be stated in terms of existence of certain Hodge tensors. The condition of "lying in $\mathbb{D}$" is more difficult to reformulate in this generality. Certainly, the *type* of the Hodge structure (i.e. the dimensions of the $V^{p,q}$) are determined by this, but non quite vice-versa (unless $G = GL_N$). In the case of Siegel's half spaces $\mathbb{H}_g^{\pm} \subset \mathrm{Hom}(\mathbb{S}, \mathrm{GSp}_{2g,\mathbb{R}})$ of section 3.2.4, it was equivalent to: type $(-1, 0), (0, -1)$ + positivity condition of the notion of polarization (cf. Lemma 3.18). In the case of a torus of the form T_F (where $\mathbb{D}$ is a point) is was equivalent to: type $(-1, 0), (0, -1)$ + fixed given CM-type (cf. Lemma 3.62).

Theorem 4.31. *Let $P_{\mathbb{C}}$ be the stabilizer in $G_{\mathbb{C}}$ of the Hodge filtration associated with h_0 (it is an algebraic group, defined over $\mathbb{C}$ only).*

1. *The association of Hodge structures and Hodge filtrations gives an* open embedding
$$\mathbb{D} = G(\mathbb{R})/\operatorname{Stab}_{G(\mathbb{R})}(h) \hookrightarrow (G/P)(\mathbb{C}).$$
G/P is in a natural way a projective complex analytic variety (called a **flag manifold***).*

2. *The functor $\mathcal{H}(G, \mathbb{D})$ is represented by $\mathbb{D}$ together with its structure as analytic manifold obtained by means of the open embedding in 1., i.e. there is are functorial bijections*
$$\mathcal{H}(G, \mathbb{D})(B) \cong \operatorname{Hom}(B, \mathbb{D}),$$
where the Hom *denotes holomorphic maps.*

3. *The complex structure obtained by means of the open embedding in 1. induces the almost complex structure given by the morphisms $u : U_1 \hookrightarrow G^{\mathrm{ad}}(\mathbb{R})$ associated with the morphisms h (cf. 4.15). In particular, the latter is integrable.*

Remark 4.32. *Actually the space $(G/P)(\mathbb{C})$ is* itself *a compact Hermitian symmetric space obtained by means of the compact form of $G^{\mathrm{ad}}_{\mathbb{R}}$ whose $\mathbb{R}$-points, by (SV2), can be given by*
$$G^{\mathrm{ad},c}(\mathbb{R}) = \{g \in G^{\mathrm{ad}}(\mathbb{C}) \mid u_0(-1) \cdot g \cdot u_0(-1) = \overline{g}\}.$$
The morphism u_0 factors through the intersection of G^{ad} and $G^{\mathrm{ad},c}$ and hence defines also a morphism $u_0 : U_1 \to G^{\mathrm{ad},c}(\mathbb{R})$ and is the one associated with this compact Hermitian symmetric space by means of 4.8. In other words, we have
$$(G/P)(\mathbb{C}) \cong G^{c,\mathrm{ad}}(\mathbb{R})/K.$$

Proof of Theorem 4.31. It is clear that $\mathcal{H}(G, \mathbb{D})(\cdot)$ is isomorphic to $\mathbb{D}$ as a set. Hence is suffices to construct a morphism of functors $\mathcal{H}(G, \mathbb{D}) \to \operatorname{Hom}(-, (G/P)(\mathbb{C}))$ (holomorphic maps). It will be automatically $G(\mathbb{R})$-equivariant and consequently the induced embedding $\mathbb{D} = G(\mathbb{R})/\operatorname{Stab}_{G(\mathbb{R})}(h) \to (G/P)(\mathbb{C})$ comes from the inclusion $G(\mathbb{R}) \to G(\mathbb{C})$. In a second step, we show that this map is an open embedding (of real manifolds). Let $e_1, \dots, e_N$ be a basis of $\mathbb{C}^N$ splitting the Hodge filtration associated with h_0.
Let B be an analytic manifold and
$$\cdots \subset \mathcal{F}^{p+1} \subset \mathcal{F}^p \subset \cdots \subset V \otimes_{\mathbb{Q}} \mathcal{O}_B$$
a filtration in $\mathcal{H}(G, \mathbb{D})(B)$. Locally on an open subset U (i.e. possibly passing from B to an open cover) we may choose an $\mathcal{O}_B$-basis $f_1, \dots, f_N$ of $V \otimes_{\mathbb{Q}} \mathcal{O}_U$ splitting the filtration $\{\mathcal{F}^p\}$. This defines a morphism
$$U \to \mathrm{GL}_N(\mathbb{C})$$
which is holomorphic. Since the filtrations are conjugated this map, considered as a linear automorphism of $V \otimes_{\mathbb{Q}} \mathcal{O}_B$, maps the standard filtration to the filtration $\{\mathcal{F}^p\}$. If $\widetilde{P}$ denotes the stabilizer of the standard filtration, the image of the map composed with the projection
$$\widetilde{\mu}_U : U \to \mathrm{GL}_N(\mathbb{C})/\widetilde{P}(\mathbb{C})$$
is independent of the choice of the basis $f_1, \dots, f_N$ and hence glues to a map
$$\widetilde{\mu}_B : B \to \mathrm{GL}_N(\mathbb{C})/\widetilde{P}(\mathbb{C}).$$
Because of the conjugacy condition in the definition of $\mathcal{H}(G, \mathbb{D})(B)$ we have the following commutative diagram of maps of sets:
$$\begin{tikzcd}
B \arrow[rr, "\widetilde{\mu}_B"] \arrow[d] \arrow[dr, dashed, "\mu_B"] & & \mathrm{GL}_N(\mathbb{C})/\widetilde{P}(\mathbb{C}) \\
\mathbb{D} = G(\mathbb{R})/\operatorname{Stab}_{G(\mathbb{R})}(h) \arrow[r, hook] & G(\mathbb{C})/P(\mathbb{C}) \arrow[ur, hook] &
\end{tikzcd}$$

and consequently there is an induced holomorphic map μ_B which we define to be the image of the filtration $\{\mathcal{F}_p\}$ in $\operatorname{Hom}(B, (G/P)(\mathbb{C}))$. Note that the map $\mathbb{D} \to G(\mathbb{C})/P(\mathbb{C})$ is injective because the Hodge filtration determines the Hodge structure.
For the existence of the projective variety (flag variety) G/P, defined over $\mathbb{C}$, with $(G/P)(\mathbb{C}) = G(\mathbb{C})/P(\mathbb{C})$ we refer to the literature on algebraic groups.
To show that $G(\mathbb{R})/\operatorname{Stab}_{G(\mathbb{R})}(h) \to G(\mathbb{C})/P(\mathbb{C})$ is *open*, we are reduced to show that the map on tangent spaces

$$\mathfrak{g}_{\mathbb{R}}/\mathfrak{k} \to \mathfrak{g}_{\mathbb{C}}/\mathfrak{p} \tag{31}$$

is an isomorphism. Here $\mathfrak{k}$, by slight abuse of notation, denotes the Lie algebra of $\operatorname{Stab}_{G(\mathbb{R})}(h)$, not of K, which may differ if $G \neq G^{\mathrm{ad}}$.
The representation $\operatorname{Ad} \circ h_0$ of $\mathbb{S}$ on $\mathfrak{g}_{\mathbb{R}}$ induces a decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{-1,1}$$

by (SV1). Let us denote the complex dimension of $\mathfrak{g}^{-1,1}$ (which agrees with that of $\mathfrak{g}^{1,-1}$) by r and the dimension of $\mathfrak{g}^{0,0}$ by k. The three spaces act on the Hodge decomposition on $V_{\mathbb{C}}$ by

$$V^{p,q} \to V^{p-1,q+1}, \qquad V^{p,q} \to V^{p,q} \qquad V^{p,q} \to V^{p+1,q-1}$$

because the Lie algebra action

$$\mathrm{d}\,\rho : \operatorname{Lie}(G) \otimes V \to V$$

is a morphism of Hodge structures. It follows

$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{0,0}$$

Furthermore, we have

$$\mathfrak{k} = \mathfrak{g}_{\mathbb{R}} \cap \mathfrak{g}^{0,0}.$$

Now we count dimensions

Lie algebra or space	$\dim_{\mathbb{R}}$
$\mathfrak{g}_{\mathbb{R}}$	$k+2r$
$\mathfrak{g}_{\mathbb{C}}$	$2(k+2r)$
$\mathfrak{k}$	k
$\mathfrak{p}$	$2(k+r)$
$\mathfrak{g}_{\mathbb{R}}/\mathfrak{k}$	$2r$
$\mathfrak{g}_{\mathbb{C}}/\mathfrak{p}$	$2r$

Therefore the dimensions of source and target of the morphism (31) agree and the morphism is an open embedding. Note that this shows that we have a canonical isomorphism $T_0(\mathbb{D}) \cong \operatorname{Lie}(G)^{-1,1}$ as complex vector spaces. From this also 3. follows, as $\operatorname{Lie}(G)^{-1,1}$ is also the eigenspace of the action of u_0 for the (identity) character where $z \in U_1$ acts by multiplication by z. $\square$

4.33. The preceding theorem explains well the interpretation of $\mathbb{D}$ as set of Hodge structures. Similarly we get an interpretation of quotients like

$$G(\mathbb{Q})\backslash\mathbb{D} \times G(\mathbb{A}_f)/K$$

as parametrizing manifolds of *isomophism classes* of rational Hodge structures with abstract level structure as we will explain now. We will impose a further condition on the Shimura datum $(G, \mathbb{D})$:

(SV5) $Z(\mathbb{Q})$ is discrete in $Z(\mathbb{A}_f)$,

where Z is the center of the reductive group G.

Lemma 4.34. *If (SV5) holds then $G(\mathbb{Q})$ acts freely and properly discontinously on $\mathbb{D} \times G(\mathbb{A}_f)/K$ for all sufficiently small compact open subgroups K.*

Proof. If K is sufficiently small one can show that it, and its projection in $G^{\mathrm{ad}}(\mathbb{A}_f)$, are torsion free. We assume this.
The stabilizer groups at $[h,\gamma]$ are given by the intersection (in $G(\mathbb{A})$)

$$G(\mathbb{Q}) \cap \mathrm{Stab}_{G(\mathbb{R})}(h) \cdot {}^{\gamma}K$$

Since the projection K_∞ of $\mathrm{Stab}_{G(\mathbb{R})}(h)$ along $p: G(\mathbb{R}) \to G^{\mathrm{ad}}(\mathbb{R})$ is compact it follows that

$$G^{\mathrm{ad}}(\mathbb{Q}) \cap K_\infty \cdot p({}^{\gamma}K)$$

is finite and hence trivial because $p({}^{\gamma}K)$ is torsion free. Now we have

$$\begin{aligned} &\ker(G(\mathbb{Q}) \cap (\mathrm{Stab}_{G(\mathbb{R})}(h) \cdot {}^{\gamma}K) \to G^{\mathrm{ad}}(\mathbb{Q}) \cap (K_\infty \cdot p({}^{\gamma}K)) \\ &= Z(\mathbb{Q}) \cap Z(\mathbb{R}) \cdot (K \cap Z(\mathbb{A}_f)) \end{aligned}$$

Since $Z(\mathbb{Q})$ is discrete in $K \cap Z(\mathbb{A}_f)$ this is also finite and trivial if K is torsion-free. The fact that $G^{\mathrm{ad}}(\mathbb{Z})$ is discrete in $G^{\mathrm{ad}}(\mathbb{R})$ implies that the action is properly discontinuously. □

Proposition 4.35. *If (SV1) and (SV5) hold and K is sufficiently small, we have*

$$\mathrm{Hom}(B, G(\mathbb{Q})\backslash\mathbb{D} \times G(\mathbb{A}_f)/K) \cong \left\{ \begin{array}{rl} \mathcal{L} \to B & \textit{local system with } G\textit{-structure } \alpha \\ \{\mathcal{F}^p\} & \textit{filtration of } \mathcal{L} \otimes_{\mathbb{Q}} \mathcal{O}_B \textit{ s.t.} \\ & \{\alpha(\mathcal{F}^p)\} \textit{ lies in } H(G,\mathbb{D}) \textit{ locally} \\ \xi \in \mathrm{Hom}((\mathbb{A}_f^N)_B, \mathcal{L}_{\mathbb{A}_f})/K & \textit{(homomorphisms of } G\textit{-structures)} \end{array} \right\}$$

In the last line $\mathrm{Hom}((\mathbb{A}_f^N)_B, \mathcal{L}_{\mathbb{A}_f})/K$ denotes the quotient sheaf of the sheaf $\mathrm{Hom}((\mathbb{A}_f^N)_B, \mathcal{L}_{\mathbb{A}_f})$ modulo K. In other words ξ consists of isomorphisms of G-structures $\xi_{U_i} : (\mathbb{A}_f^N)_{U_i} \to \mathcal{L}_{\mathbb{A}_f}|_{U_i}$ on a cover $\{U_i\}$ of B such that they are equal on overlaps *modulo K*. As an exercise one should think about this in the special case $(G,\mathbb{D}) = (\mathrm{GSp}_{2g}, \mathbb{D})$. One gets precisely the moduli problem of section 3 for families of complex tori $\pi : X \to B$ with polarization and K-level-structure, setting $\mathcal{L} := R^1\pi_*\mathbb{Q}$ and noting that $\mathcal{L}_{\mathbb{A}_f}$ is canonically isomorphic to the (family of) rational Tate modules $V(X)$.

Proof (sketch) of Proposition 4.35. Locally on U the local system $\mathcal{L}$ is trivial and choosing a representative of the G-structure:

$$\alpha_U : \mathcal{L}|_U \xrightarrow{\sim} (\mathbb{Q}^N)_U$$

yields a filtration

$$\{\alpha_U(\mathcal{F}^p)\}$$

of $\mathbb{Q}^N \otimes \mathcal{O}_U$ and therefore, by Theorem 4.31, a holomorphic map

$$\mu : U \to \mathbb{D}.$$

Furthermore

$$\alpha_U \circ \xi_U \in G(\mathbb{A}_f)/K$$

is a well-defined class. We get a morphism

$$\begin{array}{rcl} U & \to & \mathbb{D} \times G(\mathbb{A}_f)/K \\ z & \mapsto & [\alpha_U(z), \alpha_U \circ \xi_U] \end{array}$$

which is, by construction, globally well-defined modulo $G(\mathbb{Q})$.
In the other direction, a morphism

$$\widetilde{\mu} : U \to G(\mathbb{Q})\backslash\mathbb{D} \times G(\mathbb{A}_f)/K$$

is locally liftable to a morphism

$$\mu : U \to \mathbb{D} \times G(\mathbb{A}_f)/K.$$

(For this it is essential, that $G(\mathbb{Q})$ acts freely and properly discontinuously!) μ corresponds to a filtration on $\mathbb{Q}^N \otimes \mathcal{O}_U$ and a level structure on $\mathbb{A}_f^N$. These can be glued to yields the required local system. □

4.7 Reinterpretation of (SV1) and (SV2) in terms of the parametrized Hodge structures

4.36. We motivated the axioms (SV1) and (SV2) by the structure of Hermitian symmetric domains. Starting with a datum of reductive group G and an arbitrary conjugacy class $\mathbb{D} \subset \mathrm{Hom}(\mathbb{S}, G)$, on the other hand, we always get a family of Hodge structures on $\mathbb{D}$ for every chosen representation $\rho : G \to \mathrm{GL}_N$. We again assume that the representation is faithful and that the Hodge structures are of a fixed weight n.
We want to translate (SV1) and (SV2) (which implied that $\mathbb{D}$ is a Hermitian symmetric domain) in terms of these families. Checking the proof, one can easily see that the statements Proposition 4.31, 1., 2., remain true without assuming (SV1) and (SV2):

1. The association of Hodge structures and Hodge filtrations gives an *open embedding*
$$\mathbb{D} = G(\mathbb{R})/\operatorname{Stab}_{G(\mathbb{R})}(h) \hookrightarrow (G/P)(\mathbb{C}).$$
G/P is in a natural way a projective complex analytic variety (called a **flag manifold**).
2. The functor $\mathcal{H}(G, \mathbb{D})$ is represented by $\mathbb{D}$ together with its structure as analytic manifold obtained by means of the open embedding in 1., i.e. there is are functorial bijections
$$\mathcal{H}(G, \mathbb{D})(B) \cong \mathrm{Hom}(B, \mathbb{D}),$$
where the Hom denotes holomorphic maps.

Only the assertion 3., that an associated morphism $u_0 : U_1 \to \mathrm{Aut}(\mathbb{D})$ exists that acts on the tangent space $T_0(\mathbb{D})$ by the character z will usually fail.

Definition 4.37. *Let $V_\mathbb{Q}$ be a $\mathbb{Q}$-vector space with Hodge structure $V_\mathbb{C} = \bigoplus_{p+q=n} V^{p,q}$ of weight n. A form*
$$\phi \in (V_\mathbb{Q}^* \otimes V_\mathbb{Q}^*)^{-n,-n}$$
is called a **polarization** *of V if the form*
$$\langle v, w\rangle := \phi(v, h(i)w)$$
is symmetric and positive definite.

Remark 4.38. *1. For a Hodge structure of weight n the assertion*
$$\phi \in (V_\mathbb{Q}^* \otimes V_\mathbb{Q}^*)^{-n,-n}$$
is equivalent to ϕ being invariant under $h(\mathbb{U}_1)$ where we consider $\mathbb{U}_1$ as subgroup of $\mathbb{S}$.

2. The calculation
$$\phi(v, w) = \phi(h(i)v, h(i)w) = \phi(w, h(i)^2 v) = (-1)^n \phi(w, v)$$
therefore shows that a polarization is alternating if n is odd and symmetric, if n is even.

Proposition 4.39. *If all H.S. in $\mathbb{D}$ (w.r.t. ρ) are of weight n and (SV2) holds then there exists $\phi \in V_\mathbb{R}^* \otimes V_\mathbb{R}^*$ s.t. all $h \in \mathbb{D}^+$ are polarized w.r.t. ϕ.*

Proof. See [4, Proposition 1.1.14., (iii)] for details. Let $\widetilde{G}$ be the compact form of $G_\mathbb{R}^{\mathrm{der}}$. Since $\widetilde{G}(\mathbb{R})$ is compact, there exists a $\widetilde{G}(\mathbb{C})$-invariant positive definite Hermitian form ψ on $V_\mathbb{C}$. From this follows that
$$\psi(gv, h_0(i)^{-1}\overline{g}h_0(i)w) = \psi(v, w) \qquad \forall g \in G^{\mathrm{der}}(\mathbb{C})$$
Note that the complex conjugations on $G^{\mathrm{der}}(\mathbb{C})$ and $\widetilde{G}(\mathbb{C})$ differ by the Cartan involution on G^{der} which by (SV2) can be given by conjugation with $h_0(i)$ (whether we consider G^{der} or G^{ad} does not matter because these groups differ by a finite group). Hence
$$\phi(v, w) := \psi(v, h_0(i)^{-1}w)$$

is $G^{\mathrm{der}}(\mathbb{R})$ invariant, in particular under $h_0(U_1)$, i.e. $\phi \in V_{h,\mathbb{R}}^{(-n,-n)}$ for all $h \in \mathbb{D}^+$ and $\psi(v,w) = \phi(v, h_0(i)w)$ is positive definite by construction. Here $V_{h,\mathbb{R}}$ is the corresponding real Hodge structure that h induces. Note that $G^{\mathrm{der}}(\mathbb{R})^+$ acts transitively in $\mathbb{D}^+$ hence writing $h = gh_0$

$$\phi(v, gh_0(i)g^{-1}w) = \psi(v, h_0(i)^{-1}gh_0(i)g^{-1}w) = \psi(g^{-1}v, g^{-1}w)$$

is symmetric and positive definite. Note that $G^{\mathrm{der}}(\mathbb{R})$ in general does not act transitively on $\mathbb{D}$ whence the restriction to $\mathbb{D}^+$. □

Under some restriction there is a converse to the previous proposition (cf. [4, Proposition 1.1.14., (iii)]).

Definition 4.40. *Let B be a complex manifold. A family of Hodge filtrations on a local system $\mathcal{L}$ over B satisfies* **Griffiths transversality***, if*

$$\nabla \mathcal{F}_i \subseteq \mathcal{F}_{i+1} \otimes_{\mathcal{O}_B} \Omega_B^1$$

where

$$\begin{aligned} \nabla : \mathcal{L} \otimes_{\mathbb{Q}} \mathcal{O}_B &\to \mathcal{L} \otimes_{\mathbb{Q}} \Omega_B^1 \\ x \otimes f &\mapsto x \otimes \mathbf{d}f \end{aligned}$$

is the canonical connection on the associated vector bundle $\mathcal{L} \otimes_{\mathbb{Q}} \mathcal{O}_B$. If $\mathcal{L}$ is some local system of cohomology groups of a variety ∇ is usually called the Gauss-Manin connection*.*

Remark 4.41. *If $\pi : X \to B$ is a smooth projective morphism of smooth algebraic varieties defined over $\mathbb{C}$ then the family of Hodge structures on*

$$R^i \pi_* \mathbb{Q}_X$$

satisfies Griffiths transversality and (on its primitive part) the structures are polarized.

Proposition 4.42. *Assume (SV2) holds and the H.S. in $\mathbb{D}$ are of constant weight n w.r.t. the representation ρ. Then the families of Hodge structures in $\mathcal{H}(G,\mathbb{D})(B)$ satisfy Griffiths transversality if and only if (SV1) is satisfied.*

Proof. See [8, Theorem 2.14] or [4, Proposition 1.1.14.] for details. It suffices to show this for the universal family over $\mathbb{D}$. We have then

$$\nabla_x \mathcal{F}^p \subset \mathcal{F}^{p+n}$$

for all $x \in T_{h_0}(\mathbb{D})^{n,-n}$. Furthermore

$$\mathrm{Lie}(G_{\mathbb{C}}) = T_{h_0}(\mathbb{D}) \oplus \mathrm{Lie}(K_{\infty,\mathbb{C}}) \oplus \overline{T_{h_0}(\mathbb{D})}$$

where $K_\infty = \mathrm{Stab}_{G_{\mathbb{R}}}(h_0)$. Here $T_{h_0}(\mathbb{D})$ is of type $(-1,1), (-2,2), \ldots$; $\mathrm{Lie}(K_{\infty,\mathbb{C}})$ is of type $(0,0)$; and $\overline{T_{h_0}(\mathbb{D})}$ is of type $(1,-1), (2,-2), \ldots$ (cf. the proof of Proposition 4.31). Note that $\mathrm{Ad} \circ h_0$ anyway induces a Hodge structure of weight 0 on $\mathrm{Lie}(G)$, because we assumed that the weight of the structures in $\mathbb{D}$ (w.r.t. any faithful ρ) is constant. □

4.8 Existence of algebraic models

Recall from Definiton 4.3 the notion of a Shimura datum $(G,\mathbb{D})$. A **morphism of Shimura data** $(G_1,\mathbb{D}_1) \to (G_2,\mathbb{D}_2)$ is a homomorphism of algebraic groups $\alpha : G_1 \to G_2$ (defined over $\mathbb{Q}$) such that for each $h \in \mathbb{D}_1$ the morphism $\alpha_{\mathbb{R}} \circ h$ lies in $\mathbb{D}_2$. The following theorem asserts that the associated Shimura varieties have uniquely determined algebraic models over $\mathbb{C}$ and morphisms of Shimura varieties give rise to morphisms between those.

Theorem 4.43 (Baily-Borel)**.** *For each Shimura datum $(G,\mathbb{D})$ the complex analytic spaces in the projective system*

$$\{G(\mathbb{Q})\backslash\mathbb{D} \times G(\mathbb{A}_f)/K\}_K$$

have (unique if K is sufficiently small) algebraic models $X(G,\mathbb{D})_K$, defined over $\mathbb{C}$, and the morphisms in the system (resp. the morphisms coming from the $G(\mathbb{A}_f)$-action) are morphisms of algebraic varieties. The varieties are smooth if K is sufficiently small.
Furthermore each morphism of Shimura data $(G_1,\mathbb{D}_1)\to(G_2,\mathbb{D}_2)$ induces a morphism of

$$\{X(G_1,\mathbb{D}_1)_K\}_K \to \{X(G_2,\mathbb{D}_2)_K\}_K$$

projective systems of algebraic varieties.

We will only give a very rough idea of the proof (cf. [2]). It relies on the construction of automorphic forms on the Shimura varieties. They are constructed as follows:

4.44. Consider the diagram

$$\begin{tikzcd}
 & \mathbb{D}\times G(\mathbb{A}_f)/K \arrow[dl, "\pi"'] \arrow[dr, "\iota"] & \\
G(\mathbb{Q})\backslash\mathbb{D}\times G(\mathbb{A}_f)/K & & (G/P)(\mathbb{C})
\end{tikzcd}$$

where the map ι is induced by the Borel embedding $\mathbb{D}\hookrightarrow(G/P)(\mathbb{C})$ of Proposition 4.31 and the map π is the projection. The right map is equivariant under the action of $G(\mathbb{R})$. Every representation ω of P gives rise to a $G(\mathbb{C})$-equivariant vector bundle L_ω on $(G/P)(\mathbb{C})$. Its pullback ι^*L_ω carries, in particular, a $G(\mathbb{Q})$ action and hence descends to a vector bundle

$$\mathcal{L}_\omega := G(\mathbb{Q})\backslash\iota^*L_\omega$$

called an **automorphic vector bundle**.

The group $P_{\mathbb{C}}$ decomposes as $K_{\infty,\mathbb{C}}\cdot P^+$, where P^+ is the unipotent radical and $K_{\infty,\mathbb{C}}$ is the complexification of $K_\infty=\mathrm{Stab}_{G_{\mathbb{R}}}(h)$.

Proposition 4.45. *The representation $\omega: K_{\infty,\mathbb{C}}\to\mathbb{G}_m$ given by the determinant of the (complexification of the) action of K_∞ on $T_{h_0}(\mathbb{D})$ extends to $P_{\mathbb{C}}$ setting it 0 on P^+. Let $K\subset G(\mathbb{A}_f)$ be a sufficiently small compact open subgroup and set:*

$$\widetilde{X}_K := \mathrm{Proj}\left(\bigoplus_m H^0(X_K,\mathcal{L}_\omega^{\otimes m})'\right)$$

Then there is an embedding

$$X_K\hookrightarrow\widetilde{X}_K(\mathbb{C})$$

which identifies X_K with the complex points of a Zariski-open subset.

In the proposition $H^0(X_K,-)'$ means that, if PGL_2 is a quotient of $G_{\mathbb{R}}$, additional growth conditions have to be imposed to make this space finite dimensional.

Exercise 4.46. *For the Shimura datum $(\mathrm{GL}_2,\mathbb{H}^\pm)$ we have*

$$H^0(X_K,\mathcal{L}_\omega)=\{\text{ usual modular forms of weight 2 on } X_K\}$$

The space $\widetilde{X}_K$ is called the **minimal compactification** (also Baily-Borel compactification) of X_K because of the following reason:

Proposition 4.47 (Borel)**.** *Let $\widetilde{V}$ be a smooth projective variety, defined over $\mathbb{C}$, and $V\subset\widetilde{V}$ be a Zariski open subset such that $\widetilde{V}\setminus V$ is a divisor with normal crossings. Then every morphism α of analytic manifolds lifts to a morphism $\widetilde{\alpha}$*

$$\begin{tikzcd}
\widetilde{V}(\mathbb{C}) \arrow[r, "\widetilde{\alpha}"] & \widetilde{X}_K(\mathbb{C}) \\
V(\mathbb{C}) \arrow[r, "\alpha"] \arrow[u, hook] & X_K \arrow[u, hook]
\end{tikzcd}$$

By Chow's theorem $\widetilde{\alpha}$ is automatically a morphism of algebraic varieties.

Using resolution of singularities, the following is a consequence:

1. The structure of algebraic variety on X_K (given as the Zariski open subset of $\widetilde{X}_K$) is unique.
2. Morphisms in the projective system and between Shimura varieties are automatically defined by morphisms of algebraic varieties.

Theorem 4.43 follows.

4.9 Canonical models over number fields

4.48. We have already discussed canonical models of the Shimura variety associated with $(\mathrm{GSp}_{2g}, \mathbb{H}_g^{\pm})$ in the first lectures. They were defined over $\mathbb{Q}$. For more general Shimura varieties it is not reasonable to expect models which are defined over $\mathbb{Q}$. However, as motivated already in 4.5, the adelic formulation and correct axioms of a Shimura datum allow to get a model over a naturally given field associated with the Shimura datum, the reflex field. We begin by explaining the definition of this field.

Definition 4.49. *To each $h \in \mathbb{D}$ we have an associated cocharacter of $G_{\mathbb{C}}$ defined as the composition*

$$\begin{tikzcd} \mathbb{G}_{m,\mathbb{C}} \arrow[r] \arrow[rr, bend left, "\mu_h"] & \mathbb{S}_{\mathbb{C}} = \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \arrow[r, "h"'] & G_{\mathbb{C}} \end{tikzcd}$$

where the first morphism is the inclusion of the first factor.
The conjugacy class of these morphisms is defined over a field $E(G,\mathbb{D})$ called the **reflex field***.*

The reflex field is also the natural field of definition of the flag variety G/P studied in Theorem 4.31.

Remark 4.50. *For $G = T_F \subseteq F^x$ and $\mathbb{D} = \{h\}$ corresponding to a CM-type S (see Section 3.4.4) the field $E = E(T,\{h\})$ coincides with the usual reflex field of the CM-Type $S \subset \mathrm{Hom}(F,\mathbb{C})$, i.e.*

$$\mathrm{Gal}(\overline{\mathbb{Q}}/E) = \{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \mid \sigma \circ S = S\}.$$

This follows directly taking the relation of S and h into account:

$$\begin{array}{rcl} h^* : X^*(F^x_{\mathbb{C}}) & \to & X^*(\mathbb{S}_{\mathbb{C}}) \\ \| & & \| \\ \mathbb{Z}[\mathrm{Hom}(F,\mathbb{C})] & & \mathbb{Z}\times\mathbb{Z} \\ [\sigma] & \mapsto & \begin{cases} (1,0) & \sigma \in S \\ (0,1) & \sigma \notin S \end{cases} \end{array}$$

Exercise 4.51. *The reflex field of $(\mathrm{GSp}_{2g}, \mathbb{H}_g^{\pm})$ is $\mathbb{Q}$.*

Definition 4.52. *Let $(G,\mathbb{D})$ be a Shimura datum. An algebraic model of X_K, defined over $E(G,\mathbb{D})$, is called* **canonical***, if for any morphism*

$$\iota : (T,\{h\}) \to (G,\mathbb{D})$$

of Shimura data, where T is a torus, the image of the morphism

$$T(\mathbb{Q})\backslash\{h\} \times T(\mathbb{A}_f)/K \cap T(\mathbb{A}_f) \to G(\mathbb{Q})\backslash\mathbb{D} \times G(\mathbb{A}_f)/K \cong X_K(\mathbb{C})$$

consists of points defined over $\overline{\mathbb{Q}}$ w.r.t. the model X_K, and the action of the Galois group[23] $\mathrm{Gal}(\overline{\mathbb{Q}}/E(T,h))$ *is given by*

$${}^{\sigma}[\iota\circ h, \xi] = [\iota \circ h, \iota(N_h(\mathrm{rec}(\sigma))\xi]$$

Here N_h is the reflex norm of $(T,\{h\})$ defined in 3.74.

[23] The existence of ι implies that $E(G,\mathbb{D}) \subset E(T,\{h\})$.

Remark 4.53. *We have already seen that the model given by $\mathcal{A}_{g,K}$ (solution to a moduli problem of polarized Abelian varieties) of the Shimura variety associated with* $(\mathrm{GSp}_{2g}, \mathbb{H}_g^{\pm})$ *is canonical in this sense (cf. Theorem 3.78).*

The important feature of this notion is, if we consider the whole projective system of a Shimura variety and if the individual varieties have canonical models then the transition morphisms in the projective system and also the $G(\mathbb{A}_f)$-action are automatically defined over $E(G, \mathbb{D})$. Furthermore if we are given a morphism of Shimura data $(G_1, \mathbb{D}_1) \to (G_2, \mathbb{D}_2)$ then the induced morphism on *canonical models* is automatically defined over the compostium $E(G_1, \mathbb{D}_1) \cdot E(G_2, \mathbb{D}_2)$. (If the morphism is induced by a closed embedding of algebraic groups then we have even $E(G_2, \mathbb{D}_2) \subset E(G_1, \mathbb{D}_1)$ as follows immediately from the definition of reflex field). We make this precise in the following theorem:

Theorem 4.54 (Deligne)**.** *Any morphism*

$$\rho : X(G_1, \mathbb{D}_1)_{K_1} \to X(G_2, \mathbb{D}_2)_{K_2}$$

between canonical models *which, over* $\mathbb{C}$, *is equal to a morphism of the form*

$$\begin{array}{rcl} X(G_1, \mathbb{D}_1)_{K_1}(\mathbb{C}) \cong G(\mathbb{Q})\backslash\mathbb{D} \times G(\mathbb{A}_f)/K & \to & G(\mathbb{Q})\backslash\mathbb{D} \times G(\mathbb{A}_f)/K \cong X(G_2, \mathbb{D}_2)_{K_2}(\mathbb{C}) \\ {[h, \xi K_1]} & \mapsto & [h \circ \alpha, \alpha(\xi) g K_2] \end{array}$$

for $\alpha : G_1 \to G_2$ morphism of algebraic groups, defined over $\mathbb{Q}$, s.t. $\alpha_{\mathbb{R}}(\mathbb{D}_1) \subset \mathbb{D}_2$ and $g \in G(\mathbb{A}_f)$, s.t. $\alpha(K_1) \subset g(K_2)g^{-1}$ is defined over $E(G_1, \mathbb{D}_1) \cdot E(G_2, \mathbb{D}_2)$.

Corollary 4.55.

1. *Canonical model are uniquely determined up to unique isomorphism.*
2. *The morphisms $J_{K,K'}(g)$ of the projective system plus $G(\mathbb{A}_f)$-action are defined over $E(G, \mathbb{D})$. I.e. if the individual varieties in the projective system are canonical models, then we have automatically a model of the whole Shimura variety in the sense of Definition 3.59.*
3. *The association that maps a Shimura datum to the canonical model of the Shimura variety (whole projective system) $(G, \mathbb{D}_1) \mapsto \{X(G, \mathbb{D})_K\}_K$ is functorial (provided the canonical models exist).*

Sketch of proof of Theorem 4.54.

1. There *exist* embeddings $(T, \{h\}) \to (G_1, \mathbb{D}_1)$. Take a maximal torus $\widetilde{T}_{\mathbb{R}} \subset G_{\mathbb{R}}$ over which some $h' \in \mathbb{D}_1$ factors. $\widetilde{T}_{\mathbb{R}}$ is the centralizer of a regular element $\widetilde{\lambda}$ in $\mathrm{Lie}(G_{1,\mathbb{R}})$. If λ is sufficiently close to $\widetilde{\lambda}$ and defined over $\mathbb{Q}$, its centralizer T will be defined over $\mathbb{Q}$ and $T_{\mathbb{R}}$ will be conjugated to $\widetilde{T}_{\mathbb{R}}$, say $gT_{\mathbb{R}}g^{-1} = \widetilde{T}_{\mathbb{R}}$. The morphism $h := g^{-1} \cdot h'$ will factor through $T_{\mathbb{R}}$ and hence we have an embedding $(T, h) \hookrightarrow (G_1, \mathbb{D}_1)$.
2. *Key fact:* For any finite extension L of $E(G_1, \mathbb{D}_1)$, there exists $(T, h) \hookrightarrow (G_1, \mathbb{D}_1)$ s.t. $E(T, h)$ is linearly disjoint from L. This step is quite involved and uses the Hilbert irreducibility theorem.
3. For any $h \in \mathbb{D}$ the set $\{[h, \xi] \mid \xi \in G(\mathbb{A}_f)\}$ is Zariski-dense in $X(G_1, \mathbb{D}_1)_K$. This follows ultimately from the fact that $G_1(\mathbb{Q})$ is dense in $G_1(\mathbb{R})$ for connected G.
4. For each embedding $(T, \{h\}) \to (G_1, \mathbb{D}_1)$ the morphism ρ restricted to the Zariski-dense set of points $\{[h, \xi] \mid \xi \in G(\mathbb{A}_f)\}$ is defined over $E(T, h)$, i.e. equivariant under $\mathrm{Gal}(\overline{\mathbb{Q}}, E(T, h))$:

$$\begin{tikzcd}
{[\iota \circ h, \xi]} \arrow[r, mapsto, "\sigma"] \arrow[dd, mapsto, "\rho"'] & {[\iota \circ h, \iota(N_h(\mathrm{rec}(\sigma))) \cdot \xi]} \arrow[d, mapsto, "\rho"] \\
 & {[\iota \circ h, \alpha(\iota(N_h(\mathrm{rec}(\sigma)) \cdot \xi) \cdot g]} \arrow[d, equal] \\
{[\alpha \circ \iota \circ h, \alpha(\iota(\xi)) \cdot g]} \arrow[r, mapsto] & {[\alpha \circ \iota \circ h, \alpha(\iota(N_h(\mathrm{rec}(\sigma))) \cdot \alpha(\xi) \cdot g]}
\end{tikzcd}$$

Here we applied the canonical model property for $X(G_2, \mathbb{D}_2)_{K_2}$ and the morphism $\alpha \circ \iota : (T, \{h\}) \to (G_2, \mathbb{D}_2)$.

5. From 3. and 4. follows that the morphism ρ is defined over $E(T,\{h\})$. (The Zariski dense set of points of 3. defines a Zariski dense set of points in the graph of ρ which is fixed under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}, E(T,\{h\}))$, Therefore the graph and hence ρ itself are defined over $E(T,\{h\})$.) Accordingly from 2. follows that ρ is actually defined over $E(G_1,\mathbb{D}_1)\cdot E(G_2,\mathbb{D}_2)$.

$\square$

4.10 Existence of canonical models

Definition 4.56. *We call a Shimura datum (resp. the associated Shimura variety) of* **Hodge type** *if there is an embedding $(G,\mathbb{D}) \hookrightarrow (\mathrm{GSp}_{2g}, \mathbb{H}_g^{\pm})$ into the Shimura datum studied in the beginning of the lecture.*

We know (cf. Theorem 3.78) that a *canonical* model of the Shimura variety associated with $(\mathrm{GSp}_{2g}, \mathbb{H}_g^{\pm})$ exists, namely the moduli space of Abelian varieties with polarization and level structure $\{\mathcal{A}_{g,K}\}_K$.

Remark 4.57. *Many Shimura data are of Hodge type. One can weaken the condition and merely ask roughly that there is an embedding $(G',\mathbb{D}') \hookrightarrow (\mathrm{GSp}_{2g}, \mathbb{H}_g^{\pm})$ such that $(G',\mathbb{D}')$ and $(G,\mathbb{D})$ have $(G')^{\mathrm{ad}} = G^{\mathrm{ad}}$ and $(\mathbb{D}')^+ = (\mathbb{D})^+$. These Shimura data are called of* Abelian type[24]*. For example, if G^{ad} is simple, then $(G,\mathbb{D})$ is of Abelian type if G^{ad} is of type A, B or C, whereas if G^{ad} is of type E_6 or E_7 then it is not of Abelian type. If G^{ad} is of type D then both cases can occur. The fact that Shimura varieties of Abelian type also have canonical models can be shown directly by a refinement of the method of this section.*

Lemma 4.58. *For an embedding of Shimura data $\alpha : (G_1,\mathbb{D}_1) \hookrightarrow (G_2,\mathbb{D}_2)$, i.e. such that the morphism $\alpha : G_1 \to G_2$ is a closed embedding, and each $K \subset G_1(\mathbb{A}_f)$ which is small enough, there exists $K' \subset G_2(\mathbb{A}_f)$ such that the natural morphism*

$$G_1(\mathbb{Q})\backslash\mathbb{D}_1 \times G_1(\mathbb{A}_f)/K \to G_2(\mathbb{Q})\backslash\mathbb{D}_2 \times G_2(\mathbb{A}_f)/K'$$

is a closed embedding.

For a model $X(G_2,\mathbb{D}_2)_{K'}$ of the second Shimura variety, we get accordingly an induced model $X(G_1,\mathbb{D}_1)_K$ of the first one. However, it is not clear, over which field it is defined. Analogous arguments to the proof of Theorem 4.54 using a Zariski dense set of points coming from the embedding of $(T,\{h\})$ into $(G_1,\mathbb{D}_1)$ show that the induced models given by means of the Lemma are defined over $E(G_1,\mathbb{D}_1)$ and are canonical. We get:

Theorem 4.59. *For Shimura varieties of Hodge type canonical models exist.*

5 Further topics

[1, 5–7, 9]

[24] The definition given in Milne [] is slighly more restrictive.

Contents

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