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Lecture on Shimura Varieties

Universität Regensburg — WS 2012/2013

The group $\mathrm{SL}_2(\mathbb{R})$ acts on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ by fractional linear transformations, yielding an isomorphism $\mathrm{Aut}(\mathbb{H}) \cong \mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$. Let Γ be a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ defined by congruence conditions. The quotients

$$\Gamma\backslash\mathbb{H},$$

more precisely, certain disjoint unions X of them, are the easiest examples of Shimura varieties. The following observations can be made:

1. X is an algebraic variety, *canonically* defined over $\mathbb{Q}$ (sometimes even over $\mathbb{Z}$). It is the solution to a moduli problem for elliptic curves with level structures.

2. There are canonical line bundles on the X's, whose sections are precisely the modular forms.

3. X can be *naturally* compactified.

4. There are distinguished points on X, the so called *CM-points*[1] or *special points*, where the Galois action can be described explicitly in terms of class field theory. There are sufficiently many of those to characterize the $\mathbb{Q}$-model of X uniquely.

5. There is a huge ring of correspondences, the so called Hecke algebra, acting on X and the modular forms. It can be used to reveal a deep connection between modular forms and 2-dimensional Galois representations.

All these facts have analogues for $\mathbb{H}$ replaced by an arbitrary *Hermitian symmetric domain* $\mathbb{D}$ and for Γ replaced by a subgroup of $G(\mathbb{Z})$ defined by congruence conditions. Here G is a linear algebraic group defined over $\mathbb{Z}$ (semi-simple over $\mathbb{Q}$) with a surjective homomorphism with compact kernel

$$G(\mathbb{R}) \to \mathrm{Aut}(\mathbb{D})^+.$$

We will study the general theory following Milne's notes [Milne1], and Deligne's articles [Deligne1–2]. There will be two appointments per week. One will be used for the lecture, and the other for discussing exercises, providing background material (e.g. on linear algebraic groups, Abelian varieties, class field theory, etc.). In the end, I would like to discuss the philosophy of Langlands and Rapoport, and derive Kottwitz' formula for the number of points in the reduction mod p of a Shimura variety, which is the starting point for the generalization of 5. above (construction of Galois representations as predicted by the Langlands program).

1 References

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[1] CM stands for 'complex multiplication' and means additional endomorphisms of an elliptic curve or Abelian variety

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