The arithmetic volume of Shimura varieties of orthogonal type

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Algebraic geometry

X smooth projective algebraic variety of dim d $/\mathbb{C}$.

Classical intersection theory

Chow groups:

$\mathrm{CH}^i(X) := \{Z \mid \; Z \text{ lin. comb. of irr. subvarieties of codim. } i\}/\text{rat. equiv.}$

Class map:

$$cl : \mathrm{CH}^i(X) \to H^{2i}(X, \mathbb{Z})$$

Intersection pairing:

$$\mathrm{CH}^i(X) \times \mathrm{CH}^j(X) \to \mathrm{CH}^{i+j}(X)_{\mathbb{Q}}$$

compatible (via class map) with cup product.
Degree map:

$$\deg : \mathrm{CH}^d(X) \to \mathbb{Z}$$

Algebraic geometry

$\mathcal{L}$ very ample line bundle on X.

$$\mathrm{c}_1(\mathcal{L}) := [\mathrm{div}(s)] \in \mathrm{CH}^1(X)$$

for any (meromorphic) global section s of $\mathcal{L}$.

Definition of geometric volume

$$\mathrm{vol}_{\mathcal{L}}(X) := \deg(\mathrm{c}_1(\mathcal{L})^d) \in \mathbb{Z}$$

$\mathcal{L}$ defines an embedding

$$X \hookrightarrow \mathbb{P}^N(\mathbb{C})$$

$X \cap H_1 \cap \cdots \cap H_d$ consists of $\mathrm{vol}_{\mathcal{L}}(X)$ points. (H_i's *generic* hyperplanes).

Algebraic geometry

Explanation of terminology: If $\mathcal{L}$ is equipped with a Hermitian metric h,

$$\mathrm{c}_1(\mathcal{L}, h) := \mathrm{d}\,\mathrm{d}^c \log h(s)$$

(which is independent of s) is a 2-form, with $cl(\mathrm{c}_1(\mathcal{L})) = [\mathrm{c}_1(\mathcal{L}, h)]$. Hence

$$\mathrm{vol}_{\mathcal{L}}(X) = \int_X \mathrm{c}_1(\mathcal{L}, h)^d.$$

Define a 'geometric degree' map:

$$\begin{array}{rcll} \mathrm{CH}^i(X) & \to & \mathbb{Z} & \\ Z & \mapsto & \deg(Z \cdot \mathrm{c}_1(\mathcal{L})^{d-i}) & \qquad = \displaystyle\int_Z \mathrm{c}_1(\mathcal{L}, h)^{d-i} \end{array}$$

Arakelov geometry

Let $\mathfrak{X}$ be a model of X, i.e. a regular projective scheme, flat over spec($\mathbb{Z}$) with fixed isomorphism $\mathfrak{X}_{\mathbb{C}} \simeq X$.

Arakelov intersection theory

Arithmetic Chow groups:

$$\widehat{\mathrm{CH}}^i(\mathfrak{X}) := \left\{ (\mathfrak{Z}, \mathfrak{g}) \,\middle|\, \begin{matrix} \mathfrak{Z} \text{ lin. comb. of irreducible subschemes of codim. } i \\ \mathfrak{g} \text{ a Greens current for } \mathfrak{Z}_{\mathbb{C}} \text{ on } X \end{matrix} \right\} \Big/ \begin{matrix} \text{rat. equiv.} \\ \text{trivial Greens cur.} \end{matrix}$$

Arithmetic intersection pairing:

$$\widehat{\mathrm{CH}}^i(\mathfrak{X}) \times \widehat{\mathrm{CH}}^j(\mathfrak{X}) \to \widehat{\mathrm{CH}}^{i+j}(\mathfrak{X})_{\mathbb{Q}}$$

Arithmetic degree map:

$$\widehat{\deg} : \widehat{\mathrm{CH}}^{d+1}(\mathfrak{X}) \to \mathbb{R}$$

($\mathfrak{g}$ "Greens current" means that $\mathrm{d}\,\mathrm{d}^c[\mathfrak{g}] + \delta_{\mathfrak{Z}_{\mathbb{C}}} = [\omega]$ holds true, for ω a smooth form.)

Arakelov geometry

$\mathcal{L}$ very ample line bundle on $\mathfrak{X}$ and h a smooth Hermitian metric on $\mathfrak{L}_{\mathbb{C}}$.

$$\widehat{\mathrm{c}}_1(\mathcal{L}, h) := [\mathrm{div}(s), \log h(s_{\mathbb{C}})] \in \widehat{\mathrm{CH}}^1(\mathfrak{X})$$

for any (meromorphic) global section s of $\mathcal{L}$.

Definition of arithmetic volume

$$\widehat{\mathrm{vol}}_{\mathcal{L},h}(\mathfrak{X}) := \widehat{\deg}(\widehat{\mathrm{c}}_1(\mathcal{L}, h)^{d+1}) \in \mathbb{R}$$

Define linear 'height' maps:

$$\begin{array}{rcll} \mathrm{Z}^i(\mathfrak{X}_{\mathbb{Q}}) & \to & \mathbb{R} & \\ Z & \mapsto & \widehat{\deg}((\overline{Z}, \mathfrak{g}) \cdot \widehat{\mathrm{c}}_1(\mathcal{L}, h)^{d+1-i}) - \int_X \mathfrak{g}\, \mathrm{c}_1(\mathcal{L}, h)^{d+1-i} & \text{(any } \mathfrak{g}\text{)} \\ & = & \widehat{\deg}(\widehat{\mathrm{c}}_1(\mathcal{L}|_{\overline{Z}}, h|_{\overline{Z}})^{d+1-i}) & \text{(if } \overline{Z} \text{ is a reasonable model)} \end{array}$$

For $i = d$, we recover the naive height (up to a bounded function)!

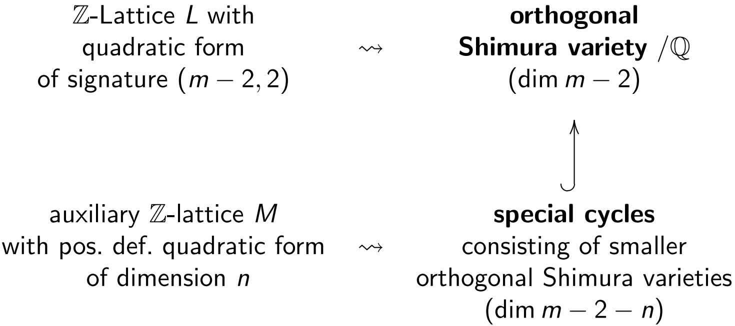
1 Algebraic geometry and Arakelov geometry

2 Orthogonal Shimura varieties

3 Main result

4 Integral models and compactifications of Shimura varieties

Orthogonal Shimura varieties ...



$\mathbb{Z}$-Lattice L with
quadratic form
of signature $(m-2,2)$

⇝

orthogonal
Shimura variety $/\mathbb{Q}$
$(\dim m-2)$

↪

auxiliary $\mathbb{Z}$-lattice M
with pos. def. quadratic form
of dimension n

⇝

special cycles
consisting of smaller
orthogonal Shimura varieties
$(\dim m-2-n)$

Structure of the special cycles reflects representations of M by L,
i.e. $\{\textit{isometries} \;\; M \hookrightarrow L\}$.

… and their geometric and arithmetic volume

	orthogonal Shimura varieties	**special cycles**
geometric volume	$\prod_\nu \lambda_\nu^{-1}(L)$	$\prod_\nu \lambda_\nu^{-1}(L)\mu_\nu(L, M, \kappa; s = 0)$
arithmetic volume	calculated in a lot of special cases (Kühn, Burgos, Bruinier, Kudla, …)	A relation to $\frac{\mathrm{d}}{\mathrm{d}\, s} \prod_\nu \mu_\nu(L, M, \kappa; s)\big\|_{s=0}$ conjectured (Kudla, Rapoport, Yang)

Here $\prod_\nu \mu_\nu(L, M, \kappa; s)$ is a Fourier coefficient of a certain Eisenstein series $E(\tau; s)$ of weight $\frac{m}{2}$ and genus n. $\lambda_p(L)$ is the volume of $\mathrm{SO}(L_{\mathbb{Z}_p})$ for almost all p.

Orthogonal Shimura varieties

Let L be a $\mathbb{Z}$-lattice with quadratic form Q_L of signature $(m-2, 2)$.

Definition: Orthogonal symmetric space

$$\mathbb{D}(L) := \{ \text{ oriented maximal negative definite subspaces of } L_{\mathbb{R}} \}$$

$\mathbb{D}(L)$ is a *complex* analytic space, homogenous under $\mathrm{SO}(L_{\mathbb{R}})$.

The corresponding orthogonal Shimura datum

There is an open (Borel) embedding:

$$\begin{array}{rcl} \mathbb{D}(L) & \hookrightarrow & \mathrm{Sh}^\vee(L)(\mathbb{C}) := \{< v > \in \mathbb{P}(L_\mathbb{C}) \mid Q_L(v) = 0\} \\ N & \mapsto & < z > \end{array}$$

determined by $N_\mathbb{C} =< z, \overline{z} >$ (taking the orientation of N into account). Associate with N the Hodge filtration

$$0 \subseteq < z > \subseteq < z >^\perp \subseteq L_\mathbb{C}$$

which corresponds to the Hodge structure of type $(-1,1),(0,0),(1,-1)$

$$L_\mathbb{C} =< z > \oplus N_\mathbb{C}^\perp \oplus < \overline{z} >$$

with associated morphism $h : \mathbb{S} \hookrightarrow \mathrm{SO}(L_\mathbb{R})$. This identifies $\mathbb{D}(L)$ with a conjugacy class of these morphisms and

$$\mathbf{O}(L) := (\mathrm{SO}(L), \mathbb{D}(L))$$

is a *Shimura datum*. It is of Abelian type.

The orthogonal Shimura variety

Definition: (Analytic) orthogonal Shimura variety

Choose a compact open subgroup $K \subseteq \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})$.

$$\mathrm{Sh}^{an}({}^K\mathbf{O}(L)) \quad := \quad [\mathrm{SO}(L_{\mathbb{Q}})\backslash \mathbb{D}(L) \times (\mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K)] = \bigcup_i [\Gamma_i \backslash \mathbb{D}(L)]\,,$$

where $\Gamma_i \subset \mathrm{SO}(L_{\mathbb{Q}})$ are certain arithmetic subgroups.

Mostly $K = \mathrm{SO}'(L_{\widehat{\mathbb{Z}}}) \subseteq \mathrm{SO}(L_{\widehat{\mathbb{Z}}})$ *(discriminant kernel)*.

Lemma

The reflex field is $\mathbb{Q}$ (if $m = \dim(L) \geq 3$).

Special cycles

Let M be another $\mathbb{Z}$-lattice, with positive definite quadratic form Q_M. For any ring R, denote:

$$\mathrm{I}(M, L)(R) := \{\alpha : M_R \to L_R \mid \alpha \text{ is an isometry } \}.$$

Definition: Special cycle

Let $\kappa \subset \mathrm{Hom}(M_{\mathbb{A}^{(\infty)}}, L_{\mathbb{A}^{(\infty)}})$ be compact open and K-stable. Then

$$\mathrm{Z}^{an}(L, M, \kappa) \quad := \sum_{K\alpha \subseteq \mathrm{I}(M,L)(\mathbb{A}^{(\infty)}) \cap \kappa} \mathrm{Sh}^{an}({}^{K'}\mathbf{O}(\alpha^{\perp}))$$

is a cycle on $\mathrm{Sh}^{an}({}^{K}\mathbf{O}(L))$.

Mostly κ will be a coset in $(L^*_{\hat{\mathbb{Z}}}/L_{\hat{\mathbb{Z}}}) \otimes M^*$ and K the discriminant kernel.

Examples

$\dim(L)$	*Witt index*	$\mathrm{Sh}({}^K\mathbf{O}(L))$	$\mathrm{Z}(L, M)$ $(\dim(M) = 1)$
2	0	CM points	$\emptyset$
3	0	Shimura curves	Heegner points
3	1	modular curves	Heegner points
4	1	Hilbert modular surfaces	Hirzebruch-Zagier cycles
5	2	Siegel threefolds	Humbert surfaces
$\vdots$	$\vdots$	$\vdots$	$\vdots$

The Weil representation and associated Eisenstein series

Assume for simplicity that $m = \dim(L)$ is even. Let M be a lattice and choose some (reference) bilinear form $\gamma_0 \in (M^* \otimes M^*)^s$.
The Weil representation ρ is a certain representation of $\mathrm{Sp}(\mathfrak{M}_{\widehat{\mathbb{Z}}})$ on $\mathbb{C}[(L^*/L) \otimes M^*]$, where $\mathfrak{M} = M^* \oplus M$ with standard symplectic form.

Definition of Eisenstein series

For $\kappa \in \mathbb{C}[(L^*/L) \otimes M^*]$ form

$$E(L, M, \kappa; \tau, s) := \sum_{g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_\infty \backslash \mathrm{Sp}(\mathfrak{M}_{\mathbb{Z}})} \frac{\det(\Im((g\tau)\gamma_0))^s}{\det(C\tau + D)^{\frac{m}{2}}} (\rho(g)\kappa)(0)$$

where $\tau \in (M \otimes M)^s_{\mathbb{C}}$ with positive definite imaginary part.

$E(L, M, \cdot; \tau, s)$ converges for $\Re(s)$ big enough, has a meromorphic continuation and is a modular form of weight $\frac{m}{2}$ for the dual of the Weil representation.

The Weil representation and associated Eisenstein series

$E(L, M, \kappa; \tau, s)$ has a Fourier expansion:

$$E(L, M, \kappa; \tau, s) = \sum_{Q \in \mathrm{Sym}^2(M^*)} \prod_{\nu} \mu_\nu(L, M^Q, \kappa; s, \Im\tau) \exp(2\pi i Q \cdot \tau)$$

(only μ_∞ actually depends on $\Im\tau$)

Actually, the special value at $s = 0$ of $\lambda^{-1}(L)E(L, M, \cdot; \tau, s)$ is a generating series for the geometric volumes of the special cycles, if one defines them for singular M in an appropriate way.

Orbit equation

Theorem (H.)

For $p \neq 2$, there exist natural continuations $\lambda_p(L; s)$ of $\lambda_p(L)$, too, such that we have

$$\lambda_p^{-1}(L; s)\mu_p(L, M, \kappa; s) = \sum_{\mathrm{SO}'(L_{\mathbb{Z}_p})\alpha \subset \mathrm{I}(M,L)(\mathbb{Q}_p) \cap \kappa_p} \lambda_p^{-1}(\alpha^\perp; s)$$

At $s = 0$ this equation specializes to

$$\mathrm{vol}(\mathrm{SO}'(L_{\mathbb{Z}_p}))^{-1} \, \mathrm{vol}(\mathrm{I}(M, L)(\mathbb{Q}_p) \cap \kappa_p) = \sum_{\mathrm{SO}'(L_{\mathbb{Z}_p})\alpha \subset \mathrm{I}(M,L)(\mathbb{Q}_p) \cap \kappa_p} \mathrm{vol}(\mathrm{SO}'(\alpha_{\mathbb{Z}_p}^\perp))^{-1}$$

where the volumes are computed w.r.t. certain *canonical* p-adic measures.

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At $s = 0$ this equation specializes to

$$\mathrm{vol}(\mathrm{I}(M, L)(\mathbb{Q}_p) \cap \kappa) = \sum_{\mathrm{SO}'(L_{\mathbb{Z}_p})\alpha \subset \mathrm{I}(M,L)(\mathbb{Q}_p) \cap \kappa_p} \frac{\mathrm{vol}(\mathrm{SO}'(L_{\mathbb{Z}_p}))}{\mathrm{vol}(\mathrm{SO}'(\alpha^\perp_{\mathbb{Z}_p}))}$$

where the volumes are computed w.r.t. certain *canonical* p-adic measures.

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At $s = 0$ this equation specializes to

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where the volumes are computed w.r.t. certain *canonical* p-adic measures.

An automorphic line bundle

The analytic construction Ξ^*

Let $\mathcal{E}$ be the restriction of the tautological bundle on $\mathbb{P}(L)$. It is an $\mathrm{SO}(L)$-bundle on $\mathrm{Sh}^\vee(\mathbf{O}(L))$.

$$\begin{tikzcd}
\mathbb{D}(L) \times (\mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K) \arrow[r] \arrow[d] & \mathrm{Sh}^\vee(\mathbf{O}(L))(\mathbb{C}) \subseteq \mathbb{P}(L)(\mathbb{C}) \\
\mathrm{SO}(L_{\mathbb{Q}})\backslash\mathbb{D}(L) \times (\mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K) &
\end{tikzcd}$$

We call $(\Xi^*\mathcal{E})_{\mathbb{C}}$ the quotient modulo $\mathrm{SO}(L_Q)$ of the pullback of $\mathcal{E}_{\mathbb{C}}$ to $\mathbb{D}(L) \times (\mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K)$.
$(\Xi^*\mathcal{E})_{\mathbb{C}}$ is an **automorphic line bundle** on $\mathrm{Sh}^{an}({}^K\mathbf{O}(L))$ and carries the quotient Ξ^*h of the Hermitian metric $h : v \mapsto [\text{const.}]\langle v, \overline{v}\rangle$

(the notation Ξ^* will be explained later)

Arithmetic theory

To define *reasonable* arithmetic volumes/heights on $\mathrm{Sh}({}^K\mathbf{O}(L))$ one needs

- integral models of compactifications of $\mathrm{Sh}({}^K\mathbf{O}(L))$,
- an *integral* Hermitian automorphic line bundle $(\Xi^*\mathcal{E}, \Xi^* h)$,
- an Arakelov theory, able to deal with the (logarithmic) singularities of $\Xi^* h$ at the boundary of the compactification.

The model of $\mathrm{Sh}({}^K\mathbf{O}(L))$ and $(\Xi^*\mathcal{E}, \Xi^* h)$ should be *canonical*, i.e. depending only on the lattice L and compatible with the embeddings used to define the special cycles.
To work with the model of $(\Xi^*\mathcal{E}, \Xi^* h)$ we needed in addition

- a *q-expansion principle* for it to detect integrality

Arithmetic theory

Main Theorem (H.)

	orthogonal Shimura varieties	**special cycles**
geometric volume w.r.t. $\mathrm{c}_1(\Xi^*\mathcal{E}_{\mathbb{C}}, \Xi^* h)$	$\prod_\nu \lambda_\nu^{-1}(L;0)$	$\prod_\nu \lambda_\nu^{-1}(L;0)\mu_\nu(L,M,\kappa;0)$
arithmetic volume w.r.t $\widehat{\mathrm{c}}_1(\Xi^*\mathcal{E}, \Xi^* h)$	$\frac{\mathrm{d}}{\mathrm{d}\,s}\prod_\nu \lambda_\nu^{-1}(L;s)\Big\|_{s=0}$ (up to $\mathbb{Q}\log(p)$ for $p^2\|4d(L)$.)	$\frac{\mathrm{d}}{\mathrm{d}\,s}\prod_\nu \lambda_\nu(L;s)\mu_\nu(L,M,\kappa;s)\Big\|_{s=0}$ (up to $\mathbb{Q}\log(p)$ for $p\|2d(L)$ and p such that $M^*_{\mathbb{Z}_p}/M_{\mathbb{Z}_p}$ is not cyclic.)

Global orbit equation

Taking the product over all ν of the local orbit equations, we get

$$\lambda^{-1}(L;s)\mu(L,M,\kappa;s)=\sum_{\mathrm{SO}'(L_{\widehat{\mathbb{Z}}})\alpha\subset \mathrm{I}(M,L)(\mathbb{A}^{(\infty)})\cap\kappa}\lambda^{-1}(\alpha^{\perp};s)$$

Note, that the sum goes over the same set of orbits, which indexes the sub Shimura varieties occurring in the special cycle.

- The value and derivative at $s=0$ of this equation express just the decomposition of the special cycle (additivity of geometric/arithmetic volume).

Borcherds theory

Theorem (Borcherds)

There is a multiplicative lift, mapping a nearly holomorphic modular form f of weight $1 - \frac{m}{2}$ for the Weil representation of L with Fourier expansion

$$\sum_{k \in \mathbb{Q}_{\gg -\infty}} a_k q^k,$$

where $a_k \in \mathbb{Z}[L^*/L]$ (sufficiently divisible), to a section F of the line bundle $(\Xi^* \mathcal{E}_{\mathbb{C}})^{\otimes l}$ on $\mathrm{Sh}^{an}({}^K\mathbf{O}(L))$.
We have

1. $\mathrm{div}(F) = \sum_{k<0} \mathrm{Z}(L, < -k >, a_k)$,
2. $l = a_0(0)$, and
3. an explicit formula for the Fourier coefficients

Interpretation of the value at 0 of the orbit equation

Serre duality and arguments of Borcherds show: f exists, iff

$$\sum_{k\in\mathbb{Q}_{\leq 0}} a_k b_{-k} = 0,$$

for each holomorphic modular form $g = \sum_{k\in\mathbb{Q}_{\gg 0}} b_k q^k$ of weight $\frac{m}{2}$ for the dual of the Weil representation.

Assume — for simplicity — that f exists, s.t. only one $a_k = \kappa$ for one $k < 0$ and $a_{k'} = 0$ for all $k' < 0, k' \neq k$.

Eisenstein series yields $\mu(L, < -k >, \kappa; 0) = a_0(0) = l$.

$$\begin{aligned} \int_{\mathrm{Sh}^{an}(\mathbf{O}(L))} \underbrace{\mathrm{d}\,\mathrm{d}^c \log h(F)}_{l\cdot \mathrm{c}_1(\Xi^*\mathcal{E}, \Xi^* h)} \mathrm{c}_1(\Xi^*\mathcal{E}, \Xi^* h)^{m-3} &= \int_{Z(L,<-k>,\kappa)_{\mathbb{C}}} \mathrm{c}_1(\Xi^*\mathcal{E}, \Xi^* h)^{m-3} \\ \mu(L, < -k >, \kappa; 0)\lambda^{-1}(L; 0) &= \sum_{\alpha} \lambda^{-1}(\alpha^{\perp}; 0) \end{aligned}$$

Interpretation of the derivative at 0 of the orbit equation

Assume f and F as before. In the thesis it is shown that

- F is an integral section of $(\Xi^*\mathcal{E})^{\otimes l}$ on the model of the compactification
- $\operatorname{div}(F)$ is equal to the Zariski closure of the special cycle + components of the boundary divisor

We have: (modulo $\mathbb{Q}\log(p)$, $p|2D$)

$$\begin{aligned}
\widehat{\deg}(\widehat{\operatorname{div}}(F)\cdot\widehat{\mathrm{c}}_1(\Xi^*\mathcal{E},\Xi^*h)^{m-2}) &= \widehat{\deg}((\widehat{\mathrm{c}}_1(\Xi^*\mathcal{E},\Xi^*h)|_{Z(L,<-k>,\kappa)})^{m-2}) \\
&\quad + \int_{\mathrm{Sh}^{an}(\mathbf{O}(L))} \log h(F)\, \mathrm{c}_1(\Xi^*\mathcal{E},\Xi^*h)^{m-2} \\
\mu(L,<-k>,\kappa;0)(\lambda^{-1})'(L;0) &= \sum_{\alpha}(\lambda^{-1})'(\alpha^\perp;0) \\
&\quad -\mu'(L,<-k>,\kappa;0)\lambda^{-1}(L;0),
\end{aligned}$$

(The integral was calculated by Kudla and also Bruinier/Kühn by different methods).

Additional problems solved in part III

- Show that the cited standard relation in Arakelov geometry applied to Borcherds products is not changed by boundary contributions — both, for the finite intersection, and for the complex analytic part — for the first, use smallness of the boundary in the Baily-Borel compactification, for the second, explicit estimates of integrals
- Construct sufficiently general Borcherds lifts with partially prescribed divisors — for this, lacunarity type results for modular forms needed.
- Prove a q-expansion principle for orthogonal modular forms using the abstract theory (part I) of canonical integral models of toroidal compactifications of *mixed* Shimura varieties
- Prove, that Fourier expansions of Borcherds lifts are integral! — for this, adelization of Borcherds product formula needed
- Investigate the Galois action of modular forms for the Weil representation.

1. Algebraic geometry and Arakelov geometry
2. Orthogonal Shimura varieties
3. Main result
4. Integral models and compactifications of Shimura varieties

Tasks

1. Theory of (toroidal) compactifications of Shimura varieties
2. Theory of mixed Shimura varieties —
   suitable for understanding the structure at the boundary of the compactifications and for getting algebraic access to Fourier expansions
3. Define automorphic vector bundles on mixed Shimura varieties and extend automorphic vector bundles to toroidal compactifications
4. Find a notion of *canonicity* for integral models of these objects, characterizing them uniquely.
5. Construct integral models

mixed Hodge structures

Definition

A *mixed Hodge structure* on a $\mathbb{Q}$-vector space L consists of an increasing weight filtration

$$0 = W_i(L) \subseteq W_{i+1}(L) \subseteq \cdots \subseteq W_{i+j}(L) = L$$

and a decreasing Hodge filtration of $L_{\mathbb{C}}$:

$$0 = F^i(L) \subseteq F^{i-1}(L) \subseteq \cdots \subseteq F^{i-k}(L) = L_{\mathbb{C}}$$

such that it defines a pure Hodge structure of weight n
on $\mathrm{gr}_W^n(L)$ for all n.

mixed Hodge structures

- occur in the (co-)homology of non-smooth and non-proper complex algebraic varieties
- those (polarized ones) of type $(-1,-1),(-1,0),(0,-1),(0,0)$ correspond to (polarized) 1-motives, i.e. homomorphisms $[X \to G]$, where X is a lattice and G is a semi-Abelian variety (easy generalization of Riemann's theorem)
- A universal family of mixed Hodge structures (with fixed filtration type and additional structure) exist and its domain $\mathbb{D}$ is homogenous under $P(\mathbb{R})U(\mathbb{C})$, where $P \subset \mathrm{GL}(L)$ is a non-reductive group over $\mathbb{Q}$ and U is the center of its unipotent radical
- We have still an embedding $h : \mathbb{D} \hookrightarrow \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$ (canonical splitting of mHs), whose images are in general not defined over $\mathbb{R}$
- There exists a 'compact' dual $\mathrm{Sh}^{\vee}$ also parametrizing *all* F-filtrations (with fixed filtration types and additional structure) without the Hodge condition. There is a Borel embedding: $\mathbb{D} \hookrightarrow \mathrm{Sh}^{\vee}$, too

mixed Shimura varteties

- Notion of mixed Shimura datum $\mathbf{X} = (P_{\mathbf{X}}, \mathbb{D}_{\mathbf{X}}, h_{\mathbf{X}})$ (Pink)
- Definition of analytic mixed Shimura variety

  $$\mathrm{Sh}^{an}({}^K\mathbf{X}) := \left[P_{\mathbf{X}}(\mathbb{Q}) \backslash \mathbb{D}_{\mathbf{X}} \times (P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K)\right]$$

  literally the same as in the pure case. (Consider this as analytic stack/orbifold)
- The map $\mathrm{Sh}({}^K\mathbf{X}) \to \mathrm{Sh}({}^K\mathbf{X}/U_{\mathbf{X}})$ is a torus-torsor!
- The notion of canonical model (Deligne) stays literally the same!

Toroidal compactifications

- For each Shimura datum **X** and maximal parabolic $Q \subset P_{\mathbf{X}}$ there is a (more mixed) Shimura datum (boundary component) **B** with $P_{\mathbf{B}} \trianglelefteq Q \subset P_{\mathbf{X}}$ and $\mathbb{D}_{\mathbf{X}} \subset \mathbb{D}_{\mathbf{B}}$.
- Subject to an additional datum Δ (a certain rational polyhedral cone decomposition) there is a compactification $\mathrm{Sh}({}^{K}_{\Delta}\mathbf{X})$ which induces a compactification $\mathrm{Sh}({}^{K'}_{\Delta'}\mathbf{B})$ for all **B** as above. These two compactifications are formally isomorphic along suitable strata in the boundary (which is a d.n.c.).
- Compactification along the torus-torsor (mixed case) is done by means of torus embeddings. This determines a rational canonical model of the toroidal compactification by requiring that the formal boundary isomorphisms decent to the reflex field.

Integral canonicity (local theory)

Let $P_{\mathbf{X}}$ now be a group scheme defined over $\mathbb{Z}_{(p)}$ (of a certain rigid type — e.g. reductive in the pure case) and call $K \subseteq P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$ admissible, if it is of the form $K = K^p \times P_{\mathbf{X}}(\mathbb{Z}_p)$ for a compact open $K^p \subset P_{\mathbf{X}}(\mathbb{A}^{(\infty,p)})$.

Let $\mathcal{O}$ be a d.v.r. above p of the reflex field E.

Form the following limit

$$\mathrm{Sh}^p(\mathbf{X})_E = \varprojlim_{K \text{ admissible}} \mathrm{Sh}({}^K\mathbf{X})_E$$

over the rational canonical models.

Extension property

In the following a *test scheme* (over R, say) is (roughly) a projective limit of etale morphisms of smooth schemes over R.

Extension property

We say that a model $\mathrm{Sh}^p(\mathbf{X})$ of $\mathrm{Sh}^p(\mathbf{X})_E$ has the extension property, if for each test scheme S over $\mathcal{O}$ and morphism

$$S_E \to \mathrm{Sh}^p(\mathbf{X})_E$$

there is a unique extension

$$S \to \mathrm{Sh}^p(\mathbf{X}).$$

A model satisfying this extension property which itself is a projective limit of etale maps of smooth schemes (over $\mathcal{O}$) is necessarily unique up to unique isomorphism and is called *integral canonical model*.

Existence of integral canonical models

Main Theorem (Kisin, Vasiu, Milne, Moonen, Deligne, H.)

Integral canonical models of mixed Shimura varieties of Abelian type exist

Structure of proof (for Hodge type).

The limit of moduli spaces of 1-motives satisfies the extension property: *Néron-Ogg-Shafarevich criterion (pure case) + Néron property (mixed case) + a theorem of Faltings*.
Extension property is inherited by closed subschemes and normalizations. This reduces the problem to showing that the normalization of the Zariski closure of a rational canonical model in the moduli space of 1-motives is smooth. This was shown by Kisin (pure case) using deep methods from p-adic Hodge theory and extended to the mixed case in my thesis. ☐

Integral models of toroidal compactifications

- Faltings & Chai constructed integral toroidal compactifications for the moduli space of 1-motives (associated with the symplectic mixed Shimura data)
- Pink showed in his thesis that rational canonical models of toroidal compactification exist over the reflex field
- Both put together give integral models of toroidal comp. of Hodge type mixed Shimura varieties and integral formal boundary isomorphisms along Zariski closures of boundary strata (gives smoothness there, too!)
- Unfortunately: Unsolved technical problem to show that this stratification is exhaustive

Integral models of toroidal compactifications

Theorem (Faltings, Chai, Pink, H.)

Integral canonical models of toroidal compactifications of (mixed) Shimura varieties of Abelian type exist and are functorial in morphisms of Shimura data, Hecke operators and refinements of Δ.

(this assumes the technical hypothesis mentioned)

Extension of automorphic vector bundles

Let $\mathbf{X} = (P_\mathbf{X}, \mathbb{D}_\mathbf{X}, h)$ be a (mixed) Shimura datum and.

Algebraization of construction of automorphic vector bundles

$$\begin{tikzcd}
{[P_\mathbf{X}(\mathbb{Q})\backslash \mathbb{D}_\mathbf{X} \times (P_\mathbf{X}(\mathbb{A}^{(\infty)})/K)]} \arrow[d, equal] & \mathbb{D}_\mathbf{X} \times (P_\mathbf{X}(\mathbb{A}^{(\infty)})/K) \arrow[l] \arrow[r, "p"] \arrow[d, hook] & h(\mathbb{D}_\mathbf{X}) \arrow[d, hook] \\
{[P_\mathbf{X}(\mathbb{Q})\backslash \mathbb{D}_\mathbf{X} \times (P_\mathbf{X}(\mathbb{A}^{(\infty)})/K)]} & P_\mathbf{X}(\mathbb{Q})\backslash \mathbb{D}_\mathbf{X} \times P_\mathbf{X}(\mathbb{C}) \times (P_\mathbf{X}(\mathbb{A}^{(\infty)})/K) \arrow[l] \arrow[r] & \mathrm{Sh}^\vee(\mathbf{X})(\mathbb{C})
\end{tikzcd}$$

Define $\mathrm{SPB}^{an}({}^K\mathbf{X}) := P_\mathbf{X}(\mathbb{Q})\backslash \mathbb{D}_\mathbf{X} \times P_\mathbf{X}(\mathbb{C}) \times (P_\mathbf{X}(\mathbb{A}^{(\infty)})/K)$ which is a $P_\mathbf{X}(\mathbf{X})(\mathbb{C})$-torsor, has a *canonical* model, called standard principal bundle.

Extension of automorphic vector bundles

Theorem (Harris, Milne, Deligne, Kisin, H.)

We have a *canonical* diagram (functorial in morphisms of (p-integral) Shimura data, Hecke operators, refinement of Δ and formal boundary maps):

$$\begin{tikzcd} \mathrm{Sh}({}^K_\Delta\mathbf{X}) & \mathrm{SPB}({}^K_\Delta\mathbf{X}) \arrow[l] \arrow[r, "p"] & \mathrm{Sh}^\vee(\mathbf{X}) \end{tikzcd}$$

of models defined over $\mathcal{O}$. The left arrow is a $P_{\mathbf{X},\mathcal{O}}$-torsor and p is $P_{\mathbf{X},\mathcal{O}}$ equivariant.

(this assumes the technical hypothesis mentioned)

One may see the diagram above as a morphism of Artin stacks:

$$\Xi : \mathrm{Sh}({}^K_\Delta\mathbf{X}) \to \left[\mathrm{Sh}^\vee(\mathbf{X})/P_{\mathbf{X},\mathcal{O}}\right]$$

such that construction of automorphic vector bundels becomes pullback.

Extension of automorphic vector bundles

- Given a $P_{\mathbf{X},\mathcal{O}}$-equivariant integral vector bundle $\mathcal{L}$ on $\mathrm{Sh}^\vee(\mathbf{X})$, with a $P_{\mathbf{X}}(\mathbb{R})U_{\mathbf{X}}(\mathbb{C})$-invariant Hermitian metric h on the image of the Borel embedding, we may form a Hermitian automorphic line bundle $(\Xi^*\mathcal{L}, \Xi^* h)$ on $\mathrm{Sh}({}^K_\Delta\mathbf{X})$ with *well-defined* Chern classes in $\widehat{\mathrm{CH}}^\bullet(\mathrm{Sh}({}^K_\Delta\mathbf{X}))_\mathbb{Q}$ (extended theory of Burgos, Kramer and Kühn)
- Compatibility with formal boundary isomorphisms gives a q-expansion priniciple for them, which can e.g. be used to prove integrality of Borcherds forms.