Fritz Hörmann

The Tate conjecture for rational K3 surfaces

Universität Freiburg — 10.02.2016

---

1 Preamble

We have seen in the seminar a way of associating with an Abelian surface A a K3 surface X such that we have an embedding $H^2(A,\mathbb{Z}) \hookrightarrow H^2(X,\mathbb{Z})$, the so-called **Kummer construction**. Similarly, we will see a construction in this lecture that associates with a K3 surface X an Abelian variety A, the so-called **Kuga-Satake construction**. We will emphasize that these constructions have a purely group theoretical provenience, namely the following. There is an exceptional isomorphism of real algebraic groups $\mathrm{Sp}_4 \cong \mathrm{Spin}(2,3)$[1] and an obvious embedding $\mathrm{Spin}(2,3) \hookrightarrow \mathrm{Spin}(2,19)$ which yields a morphism of algebraic groups

$$\mathrm{Sp}_4 \to \mathrm{Spin}(2,19)$$

which 'explains' the Kummer construction. Furthermore $\mathrm{Spin}(2,19)$ is equipped, by its definition using the Clifford algebra, with a natural 2^{21}-dimensional representation, which turns out to be symplectic, hence there is a morphism of algebraic groups

$$\mathrm{Spin}(2,19) \to \mathrm{Sp}_{2^{21}}$$

which 'explains' the Kuga-Satake construction. The problem is that I cheated a bit, because K3 surfaces are rather associated with $\mathrm{SO}(2,19)$ and there is only a 2:1 cover $\mathrm{Spin}(2,19) \to \mathrm{SO}(2,19)$. This is no problem for the Kummer construction, but as we will see, causes some indirectness in the Kuga-Satake construction. This might be seen as a hint why, up to the present day, no purely algebraic construction of the latter is known. The Hodge conjecture predicts that such a construction should exist.
In the second part of the lecture, we will show how these constructions can be used to transport the validity of the **Weil conjecture** and partial validity of the **Tate conjecture** for Abelian varieties to the respective validity for K3 surfaces. For the Weil conjectures, this idea had been used by Deligne [3] already prior to his proof for all varieties given in 1974 [4]. After Faltings proved the Tate conjecture for endomorphisms of Abelian varieties [6], the same method has been used by André [1] to obtain the Tate conjecture for K3 surfaces over $\mathbb{Q}$. More recently even the Tate conjecture for K3 over finite fields (of odd characteristic) has been proven along these lines by Madapusi Pera [7].

2 The Weil, Hodge and Tate conjectures

Given a smooth projective algebraic variety X, defined over $\mathbb{Q}$, we have the associated cohomology groups of the complex manifold $\mathcal{X} := X(\mathbb{C})$ described by X:

$$H^i(\mathcal{X},\mathbb{Q}).$$

If we instead take real, or $\mathbb{Q}_l$-coefficients, respectively, we get additional structures.

1. $$H^i(\mathcal{X},\mathbb{R})$$

   is equipped with a Hodge structure, or equivalently, with an (algebraic) action of $\mathbb{C}^*$ such that we have

   $$H^i(\mathcal{X},\mathbb{C}) = \bigoplus_{p+q=i} H(\mathcal{X})^{p,q}$$

[1] Both groups are split and simply connected, and there is an obvious isomorphism of Dynkin diagrams $\cong$.

where

$$H(\mathcal{X})^{p,q} = \{v \in H^{p+q}(\mathcal{X}, \mathbb{C}) \mid z \cdot v = z^p \overline{z}^q v\}.$$

(The complex structure on $X(\mathbb{C})$ induces an action of $\mathbb{C}^*$ on the tangent spaces which induces an action on differential forms which gives precisely this action on cohomology groups).

2.

$$H^i(\mathcal{X}, \mathbb{Q}_l)$$

is equipped with a continuous action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This is a bit harder to see. As an easy example, we discuss the case $i = 1$ for an Abelian variety A below. It is more analogous the the previous action as it may seem at first sight, because we have (as $\mathrm{Gal}(\overline{\mathbb{Q}_l}/\mathbb{Q}_l)$ modules):

$$H^i(\mathcal{X}, \mathbb{C}_l) = \bigoplus_{p+q=i} H(X)^{p,q} \otimes_{\mathbb{Q}} \mathbb{C}_l(q)$$

where the $H(X)^{p,q}$ are $\mathbb{Q}$-vector spaces that have the same dimensions as before. (Here $\mathbb{C}_l$ is the completion of an algebraic closure of $\mathbb{Q}_l$ and $\mathrm{Gal}(\overline{\mathbb{Q}_l}/\mathbb{Q}_l)$ acts on it as well.) and (q) means twisting with χ^q, where χ is the cyclotomic character. This is a result of Faltings and the starting point of p-adic Hodge theory.

If $Y \subset X$ is a subvariety of codimension i, defined over $\mathbb{Q}$, we can associate to it a cohomology class

$$[\mathcal{Y}] \subset H^{2i}(\mathcal{X}, \mathbb{Q}(i)).$$

It has the property that

$$z \cdot [\mathcal{Y}] = [\mathcal{Y}] \qquad z \in \mathbb{C}^*, \text{ resp. } z \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$$

We have then

Conjecture 2.1 (Hodge (resp. Tate) conjecture)**.** *Conversely, any class $\alpha \in H^{2i}(\mathcal{X}, \mathbb{Q}(i))$ with $z \cdot \alpha = \alpha$ for all $z \in \mathbb{C}^*$ (resp. for all $z \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$) is algebraic, that is, there are subvarieties $Y_1, \ldots, Y_n$ of codimension i of $X_{\mathbb{C}}$ (resp. X) such that*

$$\alpha = \sum_i \alpha_i [\mathcal{Y}_i]$$

with $\alpha_i \in \mathbb{Q}$.

There are only special cases of the conjecture known. For example

Theorem 2.2 (Lefschetz)**.** *The Hodge conjecture is true for $H^2(X, \mathbb{Q}(1))$ and any smooth projective variety X.*

Theorem 2.3 (Faltings)**.** *[6] The Tate conjecture is true for $H^2(A, \mathbb{Q}(1))$, where A is an Abelian variety.*

For the Galois action one has

Theorem 2.4 (Deligne; Weil conjecture[2])**.** *For each $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which is a Frobenius element[3] at $p \neq l$ we have that*

$$\mathrm{charpol}(\sigma^{-1} | H^i(\mathcal{X}, \mathbb{Q}_l))$$

is a rational polynomial whose complex eigenvalues have absolute value $p^{\frac{i}{2}}$.

Definition 2.5. *We define a* **Hodge structure, resp. Tate structure, resp. a Hodge-Tate structure**[4] *to be a $\mathbb{Q}$-vector space V together with an (algebraic) representation of $\mathbb{C}^*$ on $V_{\mathbb{R}}$, resp. a continuous representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $V_{\mathbb{Q}_l}$, resp. with both.*

Note that Hodge, Tate, and Hodge-Tate structures form $\mathbb{Q}$-linear Abelian tensor categories.

[2] This is, in fact, only a part of the original Weil conjectures and they are usually stated for varieties modulo p.

[3] Frobenius elements at p (for varying p) lie dense in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and are characterized by

$$|\sigma(x) - x^p|_p < 1 \text{ for all } x \in \overline{\mathbb{Q}} \text{ with } |x|_p \leq 1.$$

Here $|\cdot|_p$ is some extension of the p-adic absolute value to $\overline{\mathbb{Q}}$. σ is only determined by $|\cdot|_p$ up to the corresponding inertia subgroup and all $|\cdot|_p$ are conjugated under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

[4] Warning: This terminology is non-standard and usually used for Hodge-Tate representations in p-adic Hodge theory.

3 The strategy

We will show later the following theorem

Theorem 3.1 (Deligne)**.** *For each K3 X, defined over $\mathbb{Q}$, there is an Abelian variety A, defined over $\mathbb{Q}$, and an embedding of Hodge-Tate structures*

$$H^2(\mathcal{X}, \mathbb{Q})(1) \hookrightarrow \mathrm{End}(H^1(\mathcal{A}, \mathbb{Q}))$$

(A is obtained from X via the Kuga-Satake construction mentioned in the introduction.)
Using the Tate conjecture for endomorphisms of Abelian varieties (deep Theorem of Faltings) and the Weil conjecture for Abelian varieties (proven already by Weil himself) we can deduce:

Corollary 3.2. *The Tate and Weil conjectures are true for K3 surfaces.*

Proof. Both conjectures are trivial for H^0 and H^4 so the case H^2 is the only interesting one.
We use that $H^1(\mathcal{A}, \mathbb{Q}_l)$ satisfies the Weil conjectures.
Let us say that a Tate structure V is of weight n if a Frobenius acts on $V_{\mathbb{Q}_l}$ by a characteristic polynomial with rational coefficients whose roots have complex absolute value $p^{\frac{n}{2}}$. Obviously, if a Tate structure has weight n then also all of its sub-structures have weight n.
In other words $H^1(\mathcal{A}, \mathbb{Q}_l)$ has weight 1. Hence $\mathrm{End}(H^1(\mathcal{A}, \mathbb{Q}))$ has weight 0. Since, by the Theorem, $H^2(\mathcal{X}, \mathbb{Q}_l)(1)$ is a sub-Tate structure, it has weight 0 as well. Since under the cyclotomic character a Frobenius goes to multiplication by p, $H^2(\mathcal{X}, \mathbb{Q}_l)$ itself has weight 2. This is the Weil conjecture.
Now to the Tate conjecture. We use that $\mathrm{End}(H^1(\mathcal{A}, \mathbb{Q}))$ satisfies the Tate conjecture (this is *not* a property of the Hodge-Tate structure), i.e. a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant endomorphism of $H^1(\mathcal{A}, \mathbb{Q})$ is induced by a self-isogeny of A. Now let ξ be an element of $H^2(\mathcal{X}, \mathbb{Q})(1)$ which is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant. It maps to an element in $\mathrm{End}(H^1(\mathcal{A}, \mathbb{Q}))$ which is Galois invariant hence is induced by a self-isogeny of A. Hence it is also in $\mathrm{End}(H^1(\mathcal{A}, \mathbb{Q}))^{(0,0)}$ and therefore in $H^2(\mathcal{X}, \mathbb{Q})(1)^{(0,0)}$. Because the Hodge conjecture holds for divisors on K3 surfaces, we see that ξ is the class induced by a divisor. From the Galois invariance of its $\mathbb{Q}_l$-realization one can deduce that it can be even found defined over $\mathbb{Q}$. $\square$

The Tate conjecture for K3 surfaces was first proven by André [1].

4 Hodge and Tate tensors

A class in $\xi \in H^i(\mathcal{X}, \mathbb{Q})$ such that ξ is invariant under $\mathbb{C}^*$, resp. $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ tautologically forces the morphism

$$h_\infty : \mathbb{C}^* \to \mathrm{Aut}(H^i(\mathcal{X}, \mathbb{R})) \qquad h_l : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})^* \to \mathrm{Aut}(H^i(\mathcal{X}, \mathbb{Q}_l))$$

to factor through the sub-group which lets ξ fixed. Hodge, resp. Tate, structures form a nice tensor category, hence we can moreover imagine a class ξ in some space of tensors $H^i(\mathcal{X}, \mathbb{Q})^{\otimes m} \otimes (H^i(\mathcal{X}, \mathbb{Q})^*)^{\otimes n}$ which are invariant under $\mathbb{C}^*$, resp. $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Their existence will also force the morphisms h_∞, resp. h_l to factor through the subgroup fixing ξ. The smallest algebraic subgroup, defined over $\mathbb{Q}$, over which h_∞ factors is called the **Mumford-Tate group**.

Lemma 4.1. *For* any *algebraic subgroup $G \subset \mathrm{GL}_N$ there exist (a finite number of) tensors t_i such that*

$$G = \{g \in \mathrm{GL}_N \mid gt_i = t_i\}$$

It is equivalent that h_∞ factors through $G_{\mathbb{R}}$ or that all t_i are Hodge tensors.
In particular, the Mumford-Tate group is precisely the group fixed by all Hodge tensors.

A similar definition can be made for Tate structures. However, I do not know of a well-established name for the corresponding group. Conjecturally (this is a consequence of both the Hodge and Tate conjecture) it is equal to the Mumford-Tate group if we have a Hodge-Tate structure coming from geometry.

4.2. Recall that for a (polarized) Abelian variety, we considered the Hodge structure on $H^1(\mathcal{A}, \mathbb{Q})$ which is of type $(1,0),(0,1)$. The polarization induces a tensor in $\bigwedge^2 H^1(\mathcal{A}, \mathbb{Q})(1)$ which is a Hodge and Tate tensor. This implies (similar to the reasoning above) that h_∞ and h_l factor through GSp_{2g} (group of symplectic similitudes w.r.t. the form given by the polarization). Up to identifying $H^1(\mathcal{A}, \mathbb{Q})$ with a fixed symplectic vector space $\mathbb{Q}^{2g}$, all morphisms h_∞ so obtained are conjugated to each other and this set is isomorphic to Siegel's upper (and lower) half space

$$\mathbb{H}_g^\pm \subset \mathrm{Hom}(\mathbb{C}^*, \mathrm{GSp}_\mathbb{R})$$

Deligne calls the pair $(\mathrm{GSp}, \mathbb{H}_g^\pm)$ a Shimura datum. The associated Shimura variety will be a moduli space for Abelian varieties, see below.

4.3. Recall that for a (polarized) K3 surface, we considered the Hodge structure on $H^2(\mathcal{X}, \mathbb{Q})$. We will twist it with 1 for reasons that will become apparent later. Then it is of type $(1,-1),(0,0),(-1,1)$. The polarization induces a vector in $H^2(\mathcal{A}, \mathbb{Q})(1)$ which is a Hodge and Tate tensor. Similarly the quadratic form $H^2(\mathcal{A}, \mathbb{Q})(1) \times H^2(\mathcal{A}, \mathbb{Q})(1) \to H^2(\mathcal{A}, \mathbb{Q})(2) \cong \mathbb{Q}$ induces a Hoge and Tate tensor in

$$H^2(\mathcal{A}, \mathbb{Q})(1)^* \otimes H^2(\mathcal{A}, \mathbb{Q})(1))^*$$

Up to identifying $H^2(\mathcal{X}, \mathbb{Q})(1)$ with $L_\mathbb{Q}$, this implies (similar to the reasoning above) that h_∞ and h_l factor through $\mathrm{SO}(v^\perp)$ where $v^\perp \subset L$ is the orthogonal complement of the chosen vector v of length d. All morphisms h_∞ so obtained are again conjugated to each other and this set is isomorphic to the $\mathbb{D}$ considered during the seminar

$$\mathbb{D} \subset \mathrm{Hom}(\mathbb{C}^*, \mathrm{SO}(v_\mathbb{R}^\perp))$$

$$\mathbb{D} \cong \{< z > \in \mathbb{P}(L_\mathbb{C}) \mid \langle z, z\rangle = 0,\ \langle z, \overline{z}\rangle < 0,\ z \perp v\}.$$

We again get a Shimura datum $(\mathrm{SO}(v_\mathbb{Q}^\perp), \mathbb{D})$. The associated Shimura variety will be a moduli space for K3 surfaces, see below.

If we are given a Hodge or Tate structure on $\mathbb{Q}^N$ which factors through a subgroup which has a second representation

$$\begin{tikzcd}
\mathbb{C}^* \arrow[r] & G_\mathbb{R} \arrow[r, hook, "/\mathbb{Q}"] \arrow[dr, "\rho", "/\mathbb{Q}"'] & \mathrm{GL}_{N,\mathbb{R}} \\
& & \mathrm{GL}_{N',\mathbb{R}}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \arrow[r] & G_{\mathbb{Q}_l} \arrow[r, hook, "/\mathbb{Q}"] \arrow[dr, "\rho", "/\mathbb{Q}"'] & \mathrm{GL}_{N,\mathbb{Q}_l} \\
& & \mathrm{GL}_{N',\mathbb{Q}_l}
\end{tikzcd}$$

we get a different Hodge or Tate structure on $\mathbb{Q}^{N'}$. Why should such a structure be algebraic, if the first one is?

4.4. Philosophically (this would follow from the Hodge and Tate conjecture) there should be a *full* subcategory of the category of Hodge-Tate structures, containing those coming from algebraic geometry, that can be described purely algebraically: the hypothetical abelian tensor category of **pure motives**. This category would be even neutral Tannakian and hence equivalent to the category of representations of a (pro-reductive) algebraic group, the **motivic Galois group** $\mathcal{G}$. Hence a Hodge structure such that the morphism h factors

$$\begin{tikzcd}
\mathbb{C}^* \arrow[r] & G_\mathbb{R} \arrow[r, hook, "/\mathbb{Q}"] & \mathrm{GL}_{N,\mathbb{R}}
\end{tikzcd}$$

would imply that the corresponding rational representation of $\mathcal{G}$ factors:

$$\begin{tikzcd}
\mathcal{G} \arrow[r] & G \arrow[r, hook] & \mathrm{GL}_{N,\mathbb{Q}}
\end{tikzcd}$$

factors. Composing the first morphism with the *other* representation of G we get

$$\mathcal{G} \longrightarrow G \xrightarrow{\rho} \mathrm{GL}_{N',\mathbb{Q}_l}$$

hence a different representation of $\mathcal{G}$, i.e. a different motive. This construction, in contrast to the previous one, would be completely algebraic!

5 The moduli spaces of Abelian varieties, resp. K3 surfaces, as models of Shimura varieties

Let L be the K3-lattice, i.e.

$$L = E_8^{\oplus 2} \oplus U^{\oplus 3}$$

and fix a vector $v \in L$ of length $d > 0$.

Theorem 5.1. *There exists varieties $M^{K3}_{N,d}$, defined over $\mathbb{Q}$, such that we have for each variety S, defined over $\mathbb{Q}$:*

$$\mathrm{Hom}(S, M^{K3}_{N,d}) = \left\{ \begin{array}{rl} \pi : X \to S & \textit{families of K3 surfaces} \\ E \in \mathrm{NS}(X/S) & \textit{polarization of degree } d \\ \xi : R^2\pi_*^{\mathrm{et}}(\mathbb{Z}/N\mathbb{Z})_X \to (L_{\mathbb{Z}/N\mathbb{Z}})_S(1) & \textit{isomorphism (level structure)} \end{array} \right\}$$

Here ξ is supposed to be orthogonal (w.r.t. the Poincaré duality form on the L.H.S. and w.r.t. the chosen form on the R.H.S.) and to send the class of E to a fixed vector $v \in L$ of length d.

Let L' be lattice $\mathbb{Z}^{2g}$ with the symplectic form

$$\begin{pmatrix} & & & d_1 & & \\ & & & & \ddots & \\ & & & & & d_g \\ -d_1 & & & & & \\ & \ddots & & & & \\ & & -d_g & & & \end{pmatrix} \tag{1}$$

Denote $d = (d_1, \ldots, d_n)$.

Theorem 5.2. *There exists varieties $M^{AV}_{g,N,d}$, defined over $\mathbb{Q}$, such that we have for each variety S, defined over $\mathbb{Q}$:*

$$\mathrm{Hom}(S, M^{AV}_{g,N,d}) = \left\{ \begin{array}{rl} \pi : A \to S & \textit{families of Abelian surfaces} \\ E \in \mathrm{NS}(A/S) & \textit{polarization of type } d \\ \xi : R^1\pi_*^{\mathrm{et}}(\mathbb{Z}/N\mathbb{Z})_A \to (\mathbb{Z}/N\mathbb{Z})_S^{2g} & \textit{isomorphism (level structure)} \end{array} \right\}$$

Here ξ is supposed to be a symplectic similitude (w.r.t. the symplectic form (Weil pairing) with values in $(\mathbb{Z}/N\mathbb{Z})(1)$ and the chosen form on $\mathbb{Q}^{2g}$).

In both cases we won't really explain what $R^1\pi_*^{\mathrm{et}}(\mathbb{Z}/N\mathbb{Z})_A$ (resp. $R^2\pi_*^{\mathrm{et}}(\mathbb{Z}/N\mathbb{Z})_X$) is. It is a local system of $\mathbb{Z}/N\mathbb{Z}$ varieties, like defined in the next section, that gives point-wise the previously mentioned $H^1(\mathcal{A}, \mathbb{Z}/N\mathbb{Z})$ (resp. $H^2(\mathcal{X}, \mathbb{Z}/N\mathbb{Z})$) with Galois action. Its precise definition uses the derived category of etale sheaves.

6 The strategy of transport of structures

We want to explain that the completion of a Hodge structure V (parametrized by a Shimura variety) to a Hodge-Tate structure is completely encoded by specifying a model of a Shimura variety. Assume that we have a reductive group G, defined over $\mathbb{Q}$, a faithful representation $\rho : G \hookrightarrow \mathrm{GL}(V)$ and a conjugacy class $\mathbb{D} := G(\mathbb{R}) \cdot h_0 \subset \mathrm{Hom}(\mathbb{C}^*, G_{\mathbb{R}})$. Any $h \in \mathbb{D}$ specifies a Hodge structure on V (via composition with ρ). If we are given a model $\{M_K\}$, defined over $\mathbb{Q}$[5], of the projective system

$$\{G(\mathbb{Q})\backslash\mathbb{D} \times G(\mathbb{A}_f)/K\}$$

i.e. we have are isomorphisms

$$M_K(\mathbb{C}) \cong G(\mathbb{Q})\backslash\mathbb{D} \times G(\mathbb{A}_f)/K$$

that are compatible with all transition morphisms (and the action of $G(\mathbb{A}_f)$) then there is even a natural Hodge-Tate structure on V.
For this, we define a **family of Hodge-Tate structures** on M_K in the following sense:

1. A local system $\mathcal{L}_\rho$ on $M_K(\mathbb{C})$
2. A filtration
$$\cdots \subset \mathcal{F}^{p+1} \subset \mathcal{F}^p \subset \cdots \subset \mathcal{L} \otimes_{\mathbb{Q}} \mathcal{O}_{M_K(\mathbb{C})}$$
by holomorphic sub-vector bundles which gives point-wise a Hodge filtration.
3. $V_{\mathbb{Z}/l^n\mathbb{Z}}$ principal covers $\widetilde{M}_{\rho,K,l^n} \to M_K$, defined over $\mathbb{Q}$, and an isomorphism[6]
$$\left(\lim_N \widetilde{M}_{\rho,K,l^n}(\mathbb{C})\right) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathcal{L}_\rho$$

This datum is constructed as follows

1. $\mathcal{L}_\rho$ is given as the quotient of
$$V \times \mathbb{D} \times G(\mathbb{A}_f)/K$$
modulo the action of $G(\mathbb{Q})$, where $G(\mathbb{Q})$ acts via the representation ρ.
2. The filtrations are just point-wise given by the Hodge filtration associated to a $\rho \circ h$. One can show that they vary holomorphically, i.e. define holomorphic sub-vector bundles.
3. Let $V_{\mathbb{Z}}$ be a lattice in $V_{\mathbb{Q}}$ such that $V_{\widehat{\mathbb{Z}}}$ is K-stable acting by ρ (such a lattice always exists). The we define:
$$\widetilde{M}_{\rho,K,N} := \left(M_{K\cap K(N)} \times V_{\mathbb{Z}/N\mathbb{Z}}\right)/K$$
where K acts on $V_{\mathbb{Z}/N\mathbb{Z}}$ via the representation ρ.

Remark: The datum, for varying K, has the property that it is compatible under the transition maps of the projective system and $G(\mathbb{A}_f)$-equivariant.
Now given a point $x \in M_K(\mathbb{Q})$, for any K, we just get a Hodge-Tate structure

$$\mathcal{L}_{\rho,x},$$

because $\mathcal{L}_x \otimes \mathbb{C}$ carries the Hodge fitration $\{\mathcal{F}^p_x\}$ and $\mathcal{L}_x \otimes_{\mathbb{Q}} \mathbb{Q}_l$ carries the Galois action from the fiber above p of the projective system

$$\left(\widetilde{M}_{\rho,K,l^n}\right)_x \to \{x\}$$

This works the same way if $x \in M_K(F)$ to the extent that we get only a Galois representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ on $\mathcal{L}_p \otimes_{\mathbb{Q}} \mathbb{Q}_l$.
Almost tautologically, by definition of the moduli problems, we get:

[5] For general data $(G, \mathbb{D})$, the field $\mathbb{Q}$ has to be replaced by the *reflex field* of $(G, \mathbb{D})$. We will ignore this fact because all reflex fields that occur in this seminar are equal to $\mathbb{Q}$

[6] Here a lattice $V_{\mathbb{Z}} \subset V_{\mathbb{Q}}$ has to be chosen such that $V_{\widehat{\mathbb{Z}}}$ is K-stable.

Lemma 6.1. *For the model $M_K^{\mathrm{AV}_g}$ of the Shimura variety associated with $(\mathrm{GSp}_{2g}, \mathbb{H}^{\pm})$ and a point $x \in M_K^{\mathrm{AV}_g}(\mathbb{Q})$ parametrizing a pair (A, E, ξ), we have a canonical isomorphism*

$$H^1(A, \mathbb{Q}) \cong \mathcal{L}_{\mathrm{gsp},x}$$

*of Hodge-Tate structures. Here '*gsp*' denotes the standard representation of GSp_{2g} on $\mathbb{Q}^{2g}$.*

Lemma 6.2. *For the model M_K^{K3} of the Shimura variety associated with $(\mathrm{SO}(L_{d,\mathbb{Q}}), \mathbb{D})$ and a point $x \in M_K^{\mathrm{K3}}(\mathbb{Q})$ parametrizing a pair (X, E, ξ), we have a canonical isomorphism*

$$H^2(X, \mathbb{Q}(1)) \cong \mathcal{L}_{\mathrm{so},x}$$

*of Hodge-Tate structures. Here '*so*' denotes the standard representation of $\mathrm{SO}(L_d)$ on L_d.*

7 Kummer and Kuga-Satake as morphisms of Shimura data

We have the following diagram of Shimura data. The detailed definition of these maps will be given in the next section. The maps ι_{Kummer} and ι'_{Kummer} are not uniquely determined. They depend on the choice of a primitive embedding of the lattice $U(2)^{\oplus 3} \hookrightarrow L$ (such that a fixed polarization vector maps to a multiple of v).

$$\begin{tikzcd}
 & (\mathrm{GSpin}(L_d), \mathbb{D}) \arrow[r, hook, "\iota_{\mathrm{KS}}"] \arrow[d, "\pi"] & (\mathrm{GSp}_{2^{21}}, \mathbb{H}^{\pm}_{2^{20}}) \\
(\mathrm{GSp}_4, \mathbb{H}^{\pm}_g) \arrow[ur, "\iota'_{\mathrm{Kummer}}"] \arrow[r, "\iota_{\mathrm{Kummer}}"] & (\mathrm{SO}(L_d), \mathbb{D}) &
\end{tikzcd}$$

This induces morphisms of the corresponding Shimura varieties:

$$\begin{tikzcd}
 & \{\mathrm{Sh}_K(\mathrm{GSpin}(L_d), \mathbb{D})\} \arrow[r, hook, "\widetilde{\iota}_{\mathrm{KS}}"] \arrow[d, "\widetilde{\pi}"] & \{\mathrm{Sh}_K(\mathrm{GSp}_{2^{21}}, \mathbb{H}^{\pm}_{2^{20}})\} \\
\{\mathrm{Sh}_K(\mathrm{GSp}_4, \mathbb{H}^{\pm}_g)\} \arrow[ur, "\widetilde{\iota}'_{\mathrm{Kummer}}"] \arrow[r, "\widetilde{\iota}_{\mathrm{Kummer}}"] & \{\mathrm{Sh}_K(\mathrm{SO}(L_d), \mathbb{D})\} &
\end{tikzcd}$$

We equipp the Shimura varieties with the following models, all defined over $\mathbb{Q}$:

$\{\mathrm{Sh}_K(\mathrm{GSp}_{2^{21}}, \mathbb{H}^{\pm}_{2^{20}})\}$	The moduli spaces of Abelian varieties $\{M_K^{AV}\}$,
$\{\mathrm{Sh}_K(\mathrm{GSpin}(L_d), \mathbb{D})\}$	The model *induced* by the model $\{M_K^{AV}\}$ of $\{\mathrm{Sh}_K(\mathrm{GSp}_{2^{21}}, \mathbb{H}^{\pm}_{2^{20}})\}$,
$\{\mathrm{Sh}_K(\mathrm{GSp}_4, \mathbb{H}^{\pm}_2))\}$	The moduli spaces of Abelian varieties $\{M_K^{AV}\}$,
$\{\mathrm{Sh}_K(\mathrm{SO}(L_d), \mathbb{D})\}$	The moduli spaces of K3 surfaces $\{M_K^{K3}\}$.

Note that it is not clear that the *induced model* $\{\mathrm{Sh}_K(\mathrm{GSpin}(L_d), \mathbb{D})\}$ is really defined over $\mathbb{Q}$. We will assume this for the moment and come back to this question soon.

Lemma 7.1. *We have the following implication*

$$\widetilde{\pi} \textit{ is defined over } \mathbb{Q} \ \Rightarrow \textit{Theorem 3.1 (Deligne)}$$

First let's discuss

Proposition 7.2. *The induced model of $\{\mathrm{Sh}_K(\mathrm{GSpin}(L_d), \mathbb{D})\}$ w.r.t. the embedding into $\{\mathrm{Sh}_K(\mathrm{GSp}_{2^{21}}, \mathbb{H}^{\pm}_{2^{20}})\}$ is defined over $\mathbb{Q}$ and the map $\widetilde{\pi}$ is defined over $\mathbb{Q}$.*

Remark 7.3. *This could be derived from Deligne's theory [2, 5] of* **canonical models** *for Shimura varieties. That theory immediately implies that the induced model on a sub-Shimura variety of a canonical model is canonical and defined over $\mathbb{Q}$ (in this case). It follows from the theory of complex multiplication for Abelian varieties that $\{M_K^{AV}\}$ defines a canonical model. Hence we are reduced to show that also $\{M_K^{K3}\}$ is a canonical model. This can be done using the theory of complex multiplication for K3 surfaces. Then $\widetilde{\pi}$, as a morphism between canonical models, is automatically defined over $\mathbb{Q}$. We will, however, show both statements more directly using the Kummer construction that we have seen in the seminar. The method is in principle the same as in the theory of canonical models, with the only difference that the rôle of the special points is played by the Kummer embeddings.*

For this we use the following elementary observation

Lemma 7.4. *Let X, X' be varieties, defined over $\mathbb{Q}$ and let $\{Y_i\}$ be a family of subvarieties of X, all defined over $\mathbb{Q}$. If $\pi : X_{\mathbb{C}} \to X'_{\mathbb{C}}$ is a morphism with the properties*

1. *$\pi|_{Y_i} : Y_{i,\mathbb{C}} \to X'_{\mathbb{C}}$ is defined over $\mathbb{Q}$,*
2. *the union of the Y_i is Zariski dense in X.*

Then π is defined over $\mathbb{Q}$.

Proof. The conditions imply that the graph of π in $X_{\mathbb{C}} \times X'_{\mathbb{C}}$ is stable under $\mathrm{Aut}(\mathbb{C})$. Hence it, and consequently π itself, are defined over $\mathbb{Q}$. ☐

Now, we have (almost) seen in the seminar that the images of the various maps $\widetilde{\iota}_{\mathrm{Kummer}}$ lie Zariski dense in $\{\mathrm{Sh}_K(\mathrm{GSpin}(L_d), \mathbb{D})\}$. Therefore, using the Lemma, we are reduced to show that the maps

$$\{\mathrm{Sh}_K(\mathrm{GSp}_4, \mathbb{H}_g^{\pm})\} \xrightarrow{\widetilde{\iota}_{KS}\circ\widetilde{\iota}'_{\mathrm{Kummer}}} \{\mathrm{Sh}_K(\mathrm{GSp}_{2^{21}}, \mathbb{H}_{2^{20}}^{\pm})\}$$

$$\{\mathrm{Sh}_K(\mathrm{GSp}_4, \mathbb{H}_g^{\pm})\} \xrightarrow{\widetilde{\iota}_{\mathrm{Kummer}}} \{\mathrm{Sh}_K(\mathrm{SO}(L_d), \mathbb{D})\}$$

are defined over $\mathbb{Q}$. However, the modular interpretation of the first map is just given by mapping A to $A^{2^{19}}$, whereas the modular interpretation of the second map is the Kummer construction. It is a completely algebraic construction that works over any field (of char 0)[7]. Therefore the map is defined over $\mathbb{Q}$. This proves Proposition 7.2.

Proof of Lemma 7.1. By Lemma 6.2 we know that the families of Hodge-Tate-structures $\mathcal{L}_{\mathrm{so}}$ on $\{\mathrm{Sh}_K(\mathrm{SO}(L_d), \mathbb{D})\}$ associated with the standard representation 'so' gives point-wise the Hodge-Tate structure $H^2(\mathcal{X}, \mathbb{Q}(1))$ of the parametrized K3 surface. By Lemma 6.1 we know that the families of Hodge-Tate-structures $\mathcal{L}_{\mathrm{gsp}}$ on $\{\mathrm{Sh}_K(\mathrm{GSp}_{2^{21}}, \mathbb{H}_{2^{20}}^{\pm})\}$ associated with the standard representation 'gsp' give point-wise the Hodge-Tate structure $H^1(\mathcal{A}, \mathbb{Q})$ of the parametrized Abelian variety.
From the construction of the family of Hodge-Tate structures follows that, for a morphism of Shimura varieties $\{\mathrm{Sh}_K(G_1, \mathbb{D}_2)\} \to \{\mathrm{Sh}_K(G_1, \mathbb{D}_2)\}$ induced by $\alpha : G_1 \to G_2$, and *if the corresponding morphism is defined over $\mathbb{Q}$ w.r.t. the chosen models*, we have an isomorphism of families of Hodge-Tate structures:

$$\boxed{\widetilde{\alpha}^*\mathcal{L}_\rho \cong \mathcal{L}_{\rho\alpha}}$$

Now observe that we have an inclusion of representations

$$\mathrm{so} \circ \pi \hookrightarrow \mathrm{end} \circ \mathrm{gsp} \circ \iota_{KS}$$

where $\mathrm{end} : \mathrm{GL}(V) \to \mathrm{GL}(V \otimes V^*)$ is the natural representation on endomorphisms and hence an inclusion

$$\widetilde{\pi}^*\mathcal{L}_{\mathrm{so}} \cong \widetilde{\iota}_{KS}^* \,\mathrm{End}(\mathcal{L}_{\mathrm{gsp}}). \tag{2}$$

[7] One has to carefully analyse the transport of level structures at this point, however, which we won't do.

Choose a point $x \in \mathrm{Sh}_K(\mathrm{GSpin}(L_d), \mathbb{D})$ (for appropriate K) which maps under $\widetilde{\pi}$ to the point parametrizing a K2 surface X, defined over $\mathbb{Q}$. The point x maps under $\widetilde{\iota}$ to a point parametrizing an Abelian variety A, defined over some finite extension of $\mathbb{Q}$ (K needs to be small enough to have representability). This Abelian variety is called the **Kuga-Satake Abelian variety** of X.
Equation (2) evaluated at this point x yields the required embedding

$$H^2(\mathcal{X}, \mathbb{Q}(1)) \hookrightarrow \mathrm{End}(H^1(\mathcal{A}, \mathbb{Q}))$$

of Hodge-Tate structures. □

This proves Theorem 3.1 and hence completes the proof of the Tate conjecture for K3 surfaces.

8 Some details of the construction

8.1. We now sketch the definition of the maps of Shimura varieties involved:
Let $L' \cong \mathbb{Q}^4$ with symplectic form ϕ_d. The group GSp_4 acts on L' as well as on the 6 dimensional space $\wedge^2 L'$. There it fixes $\phi_d \in \wedge^2 L'$. Furthermore the natural bilinear form

$$Q : \wedge^2 L' \times \wedge^2 L' \to \wedge^4 L'$$

up to scalar. A more refined analysis shows that this construction defines an isomorphism:

$$\mathrm{GSp}_4 \xrightarrow{\sim} \mathrm{GSpin}((\phi_d)^{\perp}).$$

The lattice $(\wedge^2 L', Q)$ is isomorphic to $U^{\oplus 3}$, where U is the hyperbolic plane.
We recall the definition of $\mathrm{GSpin}(L)$:

Definition 8.2. *Let L, Q be a quadratic space We call the* ***Clifford algebra*** $\mathrm{C}(L)$ *of L an algebra with the following universal property*

$$\mathrm{Hom}_{alg}(\mathrm{C}(L), R) = \{f : \mathrm{Hom}_{\mathbb{Q}}(L, R) \mid f(v)^2 = Q(v)\}$$

for any $\mathbb{Q}$-algebra R.

The Clifford algebra of (L, Q) exists, and is obtained from the tensor algebra $\mathrm{T}(L)$ by factoring out the relation $v^2 - Q(v)$. Since the relations are homogeneous if the degree is considered mod 2, a 2-grading

$$\mathrm{C}(L) = \mathrm{C}(L)^+ \oplus \mathrm{C}(L)^-$$

survives. It is locally free of finite dimension.
From $v^2 = Q(v)$ one derives $vw + wv = \langle v, w\rangle_Q$.
We define the **main involution** on $\mathrm{C}(L)$ by

$$(v_1 \cdots v_n)' = (-v_n) \cdots (-v_1).$$

There are algebraic group $\mathrm{Spin}(L)$ and $\mathrm{GSpin}(L)$, defined over $\mathbb{Q}$, characterized by the following functors of points:

$$\mathrm{Spin}(L)(R) = \left\{g \in \mathrm{C}(L)_R^+ \mid gg' = 1,\ gLg^{-1} = L\right\}$$
$$\mathrm{GSpin}(L)(R) = \left\{g \in \mathrm{C}(L)_R^+ \mid gg' = \lambda(g),\ gLg^{-1} = L\right\}$$

where $\lambda(g) \in R^*$. They sit in a diagram of morphisms of algebraic groups with exact rows and columns:

$$\begin{tikzcd}
1 \arrow[r] & \mu_2 \arrow[r] \arrow[d, hook] & \mathrm{Spin}(L) \arrow[r] \arrow[d, hook] & \mathrm{SO}(L) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[d, "2"] & \mathrm{GSpin}(L) \arrow[r] \arrow[d, "\lambda"] & \mathrm{SO}(L) \arrow[r] & 1 \\
& \mathbb{G}_m \arrow[r, equal] & \mathbb{G}_m & &
\end{tikzcd}$$

8.3. Now recall that the Kummer construction induces an embedding

$$U(2)^{\oplus 3} \oplus O \hookrightarrow L$$

where $U(2)$ is the hyperbolic plane with quadratic form multiplied by 2. Both lattices $U(2)^{\oplus 3}$ and O are primitive, orthogonal to each other, and of discriminant 2^6. Consider the induced map (which multiplies the quadratic form by 2):

$$U^{\oplus 3} \hookrightarrow L$$

The situation may be arranged such that it maps ϕ_d to a multiple of v and hence induces an embedding

$$\mathrm{GSpin}((\phi_d)^{\perp}_{\mathbb{Q}}) \to \mathrm{GSpin}(L_{d,\mathbb{Q}})$$

which, however, does not have good integrality properties (an isometry of the lattice $U^{\oplus 3}$ usually does not extend to an isometry of L). Together with the arguments above we arrive at a morphism of Shimura data:

$$\iota'_{\mathrm{Kummer}} : (\mathrm{GSp}_4, \mathbb{H}_2^{\pm}) \to (\mathrm{GSpin}(L_{d,\mathbb{Q}}), \mathbb{D})$$

It depends on the embedding $U^{\oplus 3} \hookrightarrow L$.

8.4. Let now L be the K3 lattice and L_d the orthogonal complement of a vector v of length $d > 0$.
We first prove that for each Hodge structure on L_d of K3-type, hence described by a morphism $h : \mathbb{C}^* \to \mathrm{SO}(L_{d,\mathbb{R}})$, the morphism h factors through $\mathrm{GSpin}(L_{d,\mathbb{R}})$. Choose an orthonormal basis x_1, x_2 on the corresponding oriented positive definite plane $N \subset L_{d,\mathbb{R}}$ and let $z \in N_{\mathbb{C}}$ be an isotropic vector (determined by the orientation). Then the lift is explicitly given by

$$\begin{aligned} \widetilde{h} : \mathbb{C}^* \quad &\to \quad \mathrm{C}^+(L, \mathbb{R}) \\ w = a + bi \quad &\mapsto \quad a + bi\frac{z\overline{z} - \overline{z}z}{\langle z, \overline{z}\rangle} = a + bx_1x_2 \end{aligned}$$

It even defines a field isomorphism $\mathbb{C} \cong \mathrm{C}^+(N, \mathbb{R})$. Projected to $\mathrm{SO}(L_{d,\mathbb{R}})$ it fixes the orthogonal complement of N and gives on N a Hodge structure of type $(-1, 1)$, $(1, -1)$ where the corresponding subspaces of $N_{\mathbb{C}}$ are the isotropic subspaces $< z >$ and $< \overline{z} >$. These are precisely the Hodge structure coming from $\mathbb{D} \subset \mathrm{Hom}(\mathbb{C}^*, \mathrm{SO}(L_{d,\mathbb{R}}))$, hence we have a natural isomorphism between the conjugacy class of $\widetilde{h} : \mathbb{C}^* \to \mathrm{GSpin}(L_{d,\mathbb{R}})$ and $\mathbb{D}$. Hence we constructed a Shimura datum with a morphism:

$$\pi : (\mathrm{GSpin}(L_{d,\mathbb{Q}}), \mathbb{D}) \to (\mathrm{SO}(L_{d,\mathbb{Q}}), \mathbb{D})$$

8.5. Now we proceed to construct the claimed morphism (Kuga-Satake construction):

$$\iota_{\mathrm{KS}} : (\mathrm{GSpin}(L_{d,\mathbb{Q}}), \mathbb{D}) \to (\mathrm{GSp}(C^+(L_{d,\mathbb{Q}})), \mathbb{H}_g^{\pm})$$

Lemma 8.6. *For an element $\delta \in \mathrm{C}^+(L_d)^*$ with $\delta' = -\delta$, the form*

$$\langle x, y\rangle_\delta \mapsto \mathrm{tr}(x\delta y')$$

on $\mathrm{C}^+(L_d)$ is symplectic, unimodular, and $\mathrm{Spin}(L_d)$-invariant (resp. $\mathrm{GSpin}(L_d)$-invariant up to scalar given by λ), where these groups act by left multiplication.

The Lemma shows that the action of $\mathrm{GSpin}(L_d)$ on $C^+(L_d)$ by left multiplication induced a morphism of algebraic groups

$$\mathrm{GSpin}(L_{d,\mathbb{Q}}) \to \mathrm{GSp}(C^+(L_{d,\mathbb{Q}}), \langle\cdot,\cdot\rangle_\delta)$$

To see that this defines a map of Shimura data

$$(\mathrm{GSpin}(L_{d,\mathbb{Q}}), \mathbb{D}) \to (\mathrm{GSp}(C^+(L_{d,\mathbb{Q}})), \mathbb{H}^{\pm}_{2^{20}})$$

we take the special element $\delta := x_1x_2$ constructed from an orthonormal basis of N.

The operator in $C^+(L_{d,\mathbb{C}})$:

$$P_z^{-1,0} := \frac{z\overline{z}}{\langle z, \overline{z}\rangle},$$

resp. its complex conjugate, satisfy

$$P_z^{-1,0} + P_z^{0,-1} = \mathrm{id}, \quad (P_z^{i,j})^2 = P_z^{i,j},$$

and on $P_z^{i,j}\,\mathrm{C}^+(L_{d,\mathbb{C}})$ the morphism $\widetilde{h}$ acts as $w^{-i}\overline{w}^{-j}$. Furthermore the form $\langle\cdot, x_1x_2\cdot\rangle_\delta = \langle\cdot, h(i)\cdot\rangle_\delta = \mathrm{tr}(x\delta y'\delta') = -\mathrm{tr}(x\delta y'\delta)$ is symmetric and definite. Hence

$$\mathrm{C}^+(L_{d,\mathbb{C}}) = P_z^{-1,0}\,\mathrm{C}^+(L_{d,\mathbb{C}}) \oplus P_z^{0,-1}\,\mathrm{C}^+(L_{d,\mathbb{C}})$$

gives a Hodge structure on $C^+(L_{d,\mathbb{R}})$ which corresponds to an element in $\mathbb{H}^{\pm}_{2^{20}} \subset \mathrm{Hom}(\mathbb{C}^*, \mathrm{GSp}(C^+(L_{d,\mathbb{R}})))$.

References

[1] Y. André. On the Shafarevich and Tate conjectures for hyper-Kähler varieties. *Math. Ann.*, 305(2): 205–248, 1996.

[2] P. Deligne. Travaux de Shimura. In *Séminaire Bourbaki, 23ème année (1970/71), Exp. No. 389*, pages 123–165. Lecture Notes in Math., Vol. 244. Springer, Berlin, 1971.

[3] P. Deligne. La conjecture de Weil pour les surfaces $K3$. *Invent. Math.*, 15:206–226, 1972. ISSN 0020-9910.

[4] P. Deligne. La conjecture de Weil. I. *Inst. Hautes Études Sci. Publ. Math.*, (43):273–307, 1974.

[5] P. Deligne. Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 247–289. Amer. Math. Soc., Providence, R.I., 1979.

[6] G. Faltings. Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. *Invent. Math.*, 73(3):349–366, 1983.

[7] K. Madapusi Pera. The Tate conjecture for K3 surfaces in odd characteristic. *Invent. Math.*, 201(2): 625–668, 2015.